Problems from Math 5222 Lectures 10, 11, 12.

Problems

1. Prove that the following expressions are tensors.

✓(a) $$A^{ij}_{,l} = \frac{\partial A^{ij}}{\partial x^l} + \begin{Bmatrix} i \\ \alpha l \end{Bmatrix} A^{\alpha j} + \begin{Bmatrix} j \\ \alpha l \end{Bmatrix} A^{i\alpha}.$$

(b) $$A^i_{j,l} = \frac{\partial A^i_j}{\partial x^l} - \begin{Bmatrix} \alpha \\ jl \end{Bmatrix} A^i_\alpha + \begin{Bmatrix} i \\ \alpha l \end{Bmatrix} A^\alpha_j.$$

(c) $$A_{ij,l} = \frac{\partial A_{ij}}{\partial x^l} - \begin{Bmatrix} \alpha \\ il \end{Bmatrix} A_{\alpha j} - \begin{Bmatrix} \alpha \\ jl \end{Bmatrix} A_{i\alpha}.$$

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✓(d) $$A^r_{ijk,l} = \frac{\partial A^r_{ijk}}{\partial x^l} - \begin{Bmatrix} \alpha \\ il \end{Bmatrix} A^r_{\alpha jk} - \begin{Bmatrix} \alpha \\ jl \end{Bmatrix} A^r_{i\alpha k} - \begin{Bmatrix} \alpha \\ kl \end{Bmatrix} A^r_{ij\alpha} + \begin{Bmatrix} r \\ \alpha l \end{Bmatrix} A^\alpha_{ijk}.$$

✓ 2. Prove that ${}_a\begin{Bmatrix} k \\ ij \end{Bmatrix} - {}_b\begin{Bmatrix} k \\ ij \end{Bmatrix}$ are components of a tensor of rank three, where ${}_a\begin{Bmatrix} k \\ ij \end{Bmatrix}$ and ${}_b\begin{Bmatrix} k \\ ij \end{Bmatrix}$ are the Christoffel symbols formed from the symmetric tensors $a_{ij}(x)$ and $b_{ij}(x)$.

3. Use the formula $\frac{\partial}{\partial x^l}\left|\frac{\partial y^i}{\partial x^j}\right| = \frac{\partial^2 y^\alpha}{\partial x^l\,\partial x^\beta}\frac{\partial x^\beta}{\partial y^\alpha}\left|\frac{\partial y^i}{\partial x^j}\right|$ and the law of transformation of relative scalars of weight W to deduce formula 33.6.