Problems from Math 5222 Lectures 4, 5, and 6

Problems

✓ **1.** Given the relation $A(i, j, k)B^{jk} = C^i$, where B^{jk} is an arbitrary symmetric tensor. Prove that $A(i, j, k) + A(i, k, j)$ is a tensor. Hence deduce that, if $A(i, j, k)$ is symmetric in j and k, then $A(i, j, k)$ is a tensor.

2. Given the relation $A(i, j, k)B^{jk} = C^i$, where B^{jk} is an arbitrary skew-symmetric tensor. Prove that $A(i, j, k) - A(i, k, j)$ is a tensor. Hence, if $A(i, j, k)$ is skew-symmetric in j and k, then $A(i, j, k)$ is a tensor.

✓ **3.** If $a(i, j)\, dx^i\, dx^j$ is an invariant for an arbitrary vector dx^i, and $a(i, j)$ is symmetric, show that $a(i, j)$ is a tensor a_{ij}.

✓ **4.** If a_{ij} is a tensor, show that A^{ij}, the cofactor of a_{ij} in $|a_{ij}|$ divided by $|a_{ij}| \neq 0$, is a tensor.

5. If $\phi(x^1, \ldots, x^n)$ is a scalar, show that $\{\partial^2\phi/\partial x^i\, \partial x^j\}$ is a tensor with respect to a set of *linear* transformations of coordinates.

6. If $|a_{ij} - \lambda b_{ij}| = 0$ for $\lambda = \lambda_1$, in one set of variables, then $|a_{ij}' - \lambda b_{ij}'| = 0$ for $\lambda = \lambda_1$, in the new set of variables. In other words, the roots of the polynomial $|a_{ij} - \lambda b_{ij}|$ are invariants.

✓ **7.** Prove that a tensor with skew-symmetric components in one coordinate system has skew-symmetric components in all coordinate systems.

✓ **8.** Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric.

✓ **9.** Show that the tensor equation $a_j{}^i\lambda_i = \alpha\lambda_j$, where α is an invariant and λ_j an arbitrary vector, demands that $a_j{}^i = \delta_j{}^i\alpha$.

10. Prove directly from the law of transformation of components that symmetry of a tensor is an invariant property.

✓ **11.** The square of the element of arc ds appears in the form

$$ds^2 = g_{ij}\, dx^i\, dx^j.$$

Let T be an admissible transformation of coordinates $x^i = x^i(y^1, \ldots, y^n)$; then $ds^2 = h_{ij}\, dy^i\, dy^j$. Prove that $|g_{ij}|$ is a relative scalar of weight two. *Hint:* $h_{ij}(y) = \dfrac{\partial x^\alpha}{\partial y^i}\dfrac{\partial x^\beta}{\partial y^j} g_{\alpha\beta}(x)$, and recall the rule for multiplication of determinants.

✓ **12.** How many independent components are there in a skew-symmetric tensor of rank two?

✓ **13.** If a_{ij} is a skew-symmetric tensor and A^i is a contravariant vector, then $a_{ij}A^iA^j = 0$.

14. Prove that, if $A(i, j, k)A^iB^jC_k$ is a scalar for arbitrary vectors A^i, B^j, and C_k, then $A(i, j, k)$ is a tensor.