Mathematics 322 **Midterm Exam** **Name:**____________________

Spring 2005 (1 pt.)

This is a closed-book examination. You will have 55 minutes to complete your solutions.

1.(20 pts.) Let $\mathbf{u} \sim (1,-1,1,-1)$ and $\mathbf{v} \sim (1,1,-1,1)$ be vectors in $\mathbf{E}^4$. Compute:
(a) $\mathbf{u}\cdot\mathbf{v}$ (b) the angle between $\mathbf{u}$ and $\mathbf{v}$ (c) $\mathbf{uv}(\mathbf{w})$.

2.(20 pts.) First simplify and then carry out explicitly any implied summations in three-dimensional space:
(a) δ_i^i (b) $\delta_j^3 \delta_k^j v^k$ (c) $\varepsilon_{i2k}\delta_p^i v^k$

3.(20 pts.) Define the phrase "$\mathbf{T}$ is a second order tensor on $\mathbf{E}^3$" and give an example of such a tensor. Show that your example satisfies the definition of a second order tensor.

4.(20 pts.) Let x, y denote Cartesian coordinates in $\mathbf{E}^2$ and consider the u, v coordinate system in $\mathbf{E}^2$ defined by $x = u^2 + v^2$, $y = u^2 - v^2$. Suppose that a second order tensor $\mathbf{T}$ in $\mathbf{E}^2$ has contravariant components

$$\left[T^{ij}\right] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the u, v coordinate system at the point where $u = 2$, $\mathrm{v} = 1$. Find the component $\tilde{T}^{12}$ of $\mathbf{T}$ in the Cartesian coordinate system at this point.

5.(20 pts.) Let u^1, u^2, u^3 be a (general) coordinate system in $\mathbf{E}^3$. For this coordinate system, define:
(a) the cellar (covariant) base vectors;
(b) the reciprocal base vectors;
(c) the Christoffel symbols (of the second kind).
Also, (d) write Newton's second law $\mathbf{f} = m\mathbf{a}$ for the motion of a point-mass in the u^1, u^2, u^3 coordinate system.

Bonus. (20 pts.) Let C: $x^i(u)$ $(i = 1,2,3)$ be a smooth curve in $\mathbf{E}^3$ in a general coordinate system x^1, x^2, x^3 for $\mathbf{E}^3$. Define

$$V(i) = \frac{dx^i}{du} \quad (i = 1,2,3)$$

and

$$A(i) = \frac{d^2x^i}{du^2} \quad (i = 1,2,3)$$

in the x^1, x^2, x^3 coordinate system.

(a) Determine whether $V(1), V(2), V(3)$ form the components of a vector (first order tensor) in $\mathbf{E}^3$. Give reasons for your answer.

(b) Determine whether $A(1), A(2), A(3)$ form the components of a vector (first order tensor) in $\mathbf{E}^3$. Give reasons for your answer

Math 322 Midterm Exam

The Math 322 midterm exam on Friday, March 4 will:

1. cover material in Chapters I, II, and III of Simmonds;
2. be a closed-book examination;
3. emphasize definitions, examples, important relations, and straightforward calculations.

Here are some sample midterm exam problems to help you understand the types of questions I expect you to be able to answer.

1. Let $\mathbf{u} \sim (2,1,-2)$, $\mathbf{v} \sim (3,-6,2)$, and $\mathbf{w} \sim (4,1,7)$ be vectors in $\mathbf{E}^3$. Compute:
   (a) $\mathbf{u}\cdot\mathbf{v}$ (b) the angle between $\mathbf{u}$ and $\mathbf{v}$ (c) $\mathbf{u}\times\mathbf{v}$ (d) $\mathbf{uv}(\mathbf{w})$.
Also, (e) determine whether $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ forms a basis for $\mathbf{E}^3$.

2. First simplify and then carry out explicitly any implied summations in three-dimensional space:
   (a) $\delta^i_j v^j u_i$ (b) $\delta^2_j \delta^j_k v^k$ (c) $\delta^3_j \delta^j_1$ (d) $\varepsilon_{i3k}\delta^i_p v^k$

3. Define the phrase "$\mathbf{T}$ is a second order tensor on $\mathbf{E}^3$" and give an example of such a tensor. Show that your example satisfies the definition of a second order tensor.

4. Consider the polar coordinate system (r,θ) in $\mathbf{E}^2$ defined by $x = r\cos(\theta)$, $y = r\sin(\theta)$. Suppose that a second order tensor $\mathbf{T}$ in $\mathbf{E}^2$ has contravariant components

$$\left[T^{ij}\right] = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

in the polar coordinate system at the point where $r = 1$, $\theta = \pi/4$. Find the component $\tilde{T}^{21}$ of $\mathbf{T}$ in the Cartesian coordinate system at this point.

5. Let $\mathbf{x} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$ be a parametric representation of a smooth curve $\mathbf{C}$ in $\mathbf{E}^3$. Define (a) the unit tangent $\mathbf{T}$, (b) the principal unit normal $\mathbf{N}$, (c) the curvature κ, (d) the unit binormal $\mathbf{B}$, and (e) the torsion τ for the curve $\mathbf{C}$ at each point. Then:
   (f) Write the Frenet equations for the curve $\mathbf{C}$.
   (g) Describe the class of smooth space curves whose curvature is zero at each point.
   (h) Describe the class of smooth space curves whose torsion is zero at each point.
   (i) Describe the class of smooth space curves whose curvature and torsion are positive and constant at each point.
   (j) Describe the class of smooth space curves whose curvature is positive and constant and whose torsion is zero at each point.

6. Consider the cylindrical coordinate system (r,θ,z) in $\mathbf{E}^3$ defined by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.
(a) Compute the cellar (covariant) base vectors $\{\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\} = \{\mathbf{g}_r,\mathbf{g}_\theta,\mathbf{g}_z\}$ in cylindrical coordinates.
(b) Compute the reciprocal base vectors $\{\mathbf{g}^1,\mathbf{g}^2,\mathbf{g}^3\} = \{\mathbf{g}^r,\mathbf{g}^\theta,\mathbf{g}^z\}$ in cylindrical coordinates.

(c) Compute the roof (contravariant) components of the vector $\mathbf{v} \sim (x^2 + y^2, z, -2)$ in cylindrical coordinates.

7. Let u^1, u^2, u^3 be a (general) coordinate system in $\mathbf{E}^3$. For this coordinate system, define:
   (a) the cellar (covariant) base vectors,
   (b) the reciprocal base vectors, and
   (c) the Christoffel symbols (of the second kind).
   (d) Given that the Christoffel symbols obey

$$\tilde{\Gamma}^i_{jk} = \frac{\partial \tilde{u}^i}{\partial u^p}\left(\frac{\partial u^q}{\partial \tilde{u}^j}\frac{\partial u^r}{\partial \tilde{u}^k}\Gamma^p_{qr} + \frac{\partial^2 u^p}{\partial \tilde{u}^j \partial \tilde{u}^k}\right)$$

under a coordinate transformation

$$u^j = f^j(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3) \quad (j = 1,2,3),$$

do they transform like the components of a tensor? Why or why not?

(e) Write Newton's second law $\mathbf{f} = m\mathbf{a}$ for the motion of a point-mass in the u^1, u^2, u^3 coordinate system.

(f) Do the contravariant components of acceleration for a moving point-mass in $\mathbf{E}^3$ transform like the contravariant components of a first order tensor? Why or why not?