Math 3304 Fall 2015 Exam 1

Your printed name: Dr. Grow

Your instructor's name: ____________________

Your section (or Class Meeting Days and Time): ____________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 1 consists of this cover page, and 5 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	15	25	20	100

1. [20] Find the value of y_0 for which the solution of the initial value problem

$$ty' - y = t^2 e^{-t}, \quad y(1) = y_0 \qquad \text{(linear, first order)}$$

approaches zero as $t \to \infty$.

$y' - \frac{1}{t}y = te^{-t}$ is the standard form of the DE.

$\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t + \cancel{c}^{\,0}} = t^{-1}$ is an integrating factor.

$t^{-1}(y' - t^{-1}y) = t^{-1}(te^{-t})$

$t^{-1}y' - t^{-2}y = e^{-t}$

$\frac{d}{dt}\left[t^{-1}y\right] = e^{-t}$

$t^{-1}y = \int e^{-t}dt$

$t^{-1}y = -e^{-t} + c$

$\therefore\ y = -te^{-t} + ct$ is the general solution; here c is an arbitrary constant.

$y_0 \overset{\text{Want}}{=} y(1) = -1e^{-1} + c(1) = c - \frac{1}{e}$

$\therefore\ y_0 + \frac{1}{e} = c.$

$\therefore\ y(t) = (y_0 + \frac{1}{e})t - te^{-t}$ is the solution of the IVP.

$0 \overset{\text{Want}}{=} \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[(y_0 + \frac{1}{e})t - te^{-t}\right].$

But $\lim_{t\to\infty} te^{-t} = 0$ by l'Hospitals rule so we must have $\boxed{y_0 = -\frac{1}{e}}$ in order for $\lim_{t\to\infty} y(t) = 0$.

2. [20] Find the explicit solution of the initial value problem

$$y' = \frac{2t}{y + t^2 y} \quad \text{subject to} \quad y(0) = -2. \qquad (\text{First order, separable})$$

$\therefore \quad \frac{dy}{dt} = \frac{1}{y}\left(\frac{2t}{1+t^2}\right)$ so we separate variables and integrate.

$$\int y\,dy = \int \frac{2t\,dt}{1+t^2}$$

$$\frac{y^2}{2} = \ln(1+t^2) + c_1.$$

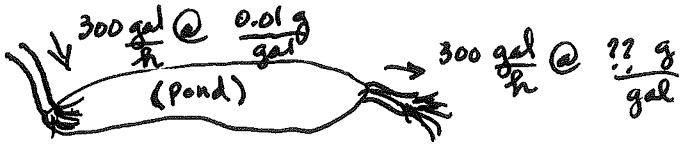
$\therefore \quad y = \pm\sqrt{c + 2\ln(1+t^2)} \qquad (c = 2c_1).$

$$-2 \overset{\text{Want}}{=} y(0) = \pm\sqrt{c + 2\ln(1)} = -\sqrt{c}.$$

$\therefore$ $c = 4$ and we must choose the negative square root.

$\boxed{y(t) = -\sqrt{4 + 2\ln(1+t^2)}}$ is the explicit solution of the IVP.

3. [15] A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical. Water containing 0.01g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Let $Q(t)$ be the amount of the chemical in the pond at time t. Write, but do not solve, an initial value problem for $Q(t)$.



Net Rate = Rate In − Rate Out

$$\frac{dQ}{dt} = \left(300\,\frac{\text{gal}}{\text{h}}\right)\left(0.01\,\frac{\text{g}}{\text{gal}}\right) - \left(300\,\frac{\text{gal}}{\text{h}}\right)\left(\frac{Q(t)\,\text{g}}{10^6\,\text{gal}}\right)$$

$V(0) = 1{,}000{,}000$ gal

$Q(0) = 0$ g

$V(t) = 1{,}000{,}000$ gal for $t > 0$.

IVP model:

$$\boxed{\frac{dQ}{dt} = 3 - \frac{3Q}{10{,}000}, \quad Q(0) = 0.}$$

(Q in grams, t in hours)

4. a) [5] Find the general solution of $y'' - 2y' + y = 0$.

$y = e^{rt}$ leads to $r^2 - 2r + 1 = 0$ so $(r-1)^2 = 0$ and $r = 1$ (mult. 2)

$\therefore$ $\boxed{y_h(t) = c_1 e^t + c_2 t e^t}$ where c_1 and c_2 are arbitrary constants.

b) [20] Find a particular solution of the differential equation $y'' - 2y' + y = \frac{e^t}{1+t^2}$. $(*)$.

(Variation of parameters)

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

where $y_1(t) = e^t$, $y_2(t) = te^t$, $W = W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = (t+1)e^{2t} - te^{2t}$
$= e^{2t}$,

$$u_1(t) = \int -\frac{y_2 g}{W}\,dt = \int \frac{-te^t \cdot e^t}{e^{2t}\cdot(1+t^2)}\,dt = \int \frac{-t\,dt}{1+t^2} = -\frac{1}{2}\ln(1+t^2) + \cancel{c_1}^{0},$$

and

$$u_2(t) = \int \frac{y_1 g}{W}\,dt = \int \frac{e^t \cdot e^t}{e^{2t}(1+t^2)}\,dt = \int \frac{dt}{1+t^2} = \text{Arctan}(t) + \cancel{c_2}^{0}.$$

Therefore $\boxed{y_p(t) = -\frac{e^t}{2}\ln(1+t^2) + te^t\,\text{Arctan}(t)}$ is a particular solution of $(*)$.

5. [20] Given that $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions for $y'' - 5y' + 6y = 0$ (**) use the method of undetermined coefficients to solve the differential equation $y'' - 5y' + 6y = 3e^{2t}$. (*)

A general solution of (*) has the form $y = y_h + y_p$ where y_h is the general solution of (**) and y_p is a particular solution of (*). From the given information, $y_h(t) = c_1 e^{3t} + c_2 e^{2t}$ where c_1 and c_2 are arbitrary constants. Since $g(t) = 3e^{2t}$, the method of undetermined coefficients states that a trial particular solution is

$$y_p = t^s(Ae^{2t})$$

where s is the smallest positive integer such that $At^s e^{2t}$ is not a solution of (**). Here A is a constant TBD. Clearly $s=1$ since e^{2t} solves (**) but te^{2t} does not. Therefore

$$y_p = Ate^{2t}$$

so $y_p' = A(2t+1)e^{2t}$ and $y_p'' = A(4t+4)e^{2t}$. We need to choose A such that

$$y_p'' - 5y_p' + 6y_p = 3e^{2t}.$$

Substituting gives

$$A(4t+4)\cancel{e^{2t}} - 5A(2t+1)\cancel{e^{2t}} + 6At\cancel{e^{2t}} = 3\cancel{e^{2t}}$$

$$\cancel{4At} + 4A - \cancel{10At} - 5A + \cancel{6At} = 3$$

$$-A = 3.$$

Consequently $y_p = -3te^{2t}$, and hence

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{2t} - 3te^{2t}} \qquad (c_1, c_2 \text{ arbitrary constants})$$

solves (*).

Math 3304

Exam I

Fall 2015

mean: 74.0

median: 89

standard deviation: 27.6

number of scores: 30

<u>Distribution of Scores</u>:

90 – 100	A	14
80 – 89	B	4
70 – 79	C	2
60 – 69	D	0
0 – 59	F	10