Math 3304 Spring 2015 Exam 3

Your printed name: Solution.

Your instructor's name: ________________

Your section (or Class Meeting Days and Time): ________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 3 consists of this cover page, 4 pages of problems containing 4 numbered problems, and 1 page of Laplace transform table.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. **Express all solution in real-valued, simplified form.**
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [25] at the beginning of a problem indicates the point value of that problem is 25. The maximum possible score on this exam is 100.

Problem	1	2	3	4	Sum
Points Earned					
Max. Points	25	25	25	25	100

1. [25] Solve the initial value problem

$$y'(t) + 3\int_0^t e^{-4(t-\tau)}y(\tau)d\tau = 0, \qquad y(0) = 1.$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\left\{\int_0^t e^{-4(t-\tau)}y(\tau)d\tau\right\} = \mathcal{L}\{0\}$$

$$\Rightarrow \left[sY(s) - y(0)\right] + 3\frac{1}{s+4}Y(s) = 0$$

$$\Rightarrow Y(s)\left[s + \frac{3}{s+4}\right] = 1$$

$$\Rightarrow Y(s)\frac{s^2+4s+3}{s+4} = 1 \Rightarrow Y(s) = \frac{s+4}{(s+1)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+3}$$

$$= \frac{(A+B)s + (3A+B)}{(s+1)(s+3)}$$

$$\therefore \left.\begin{array}{l} A+B=1 \\ 3A+B=4 \end{array}\right\} \Rightarrow \begin{array}{l} A=3/2 \\ B=-1/2 \end{array}$$

$$\Rightarrow Y(s) = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}$$

$$\therefore \boxed{y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}}$$

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2. a) [21] Solve the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 2\delta(t - \frac{\pi}{2})\sin(t); \qquad y(0) = 1, \quad y'(0) = -1.$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-\frac{\pi}{2})\sin t\}$$

$$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = 2\int_0^{\infty} e^{-st}\delta(t-\frac{\pi}{2})\sin t\,dt$$

$$\Rightarrow [s^2+2s+5]Y(s) - s + 1 - 2 = 2\left[\int_0^{\infty} e^{-st}\delta(t-\frac{\pi}{2})\sin t\,dt + \int_{-\infty}^{0} e^{-st}\delta(t-\frac{\pi}{2})\sin t\,dt\right]$$

$$\Rightarrow (s^2+2s+5)Y(s) = s+1 + 2\int_{-\infty}^{\infty} e^{-st}\delta(t-\frac{\pi}{2})\sin t\,dt \qquad (\because \delta(t-\frac{\pi}{2}) = 0 \text{ for } t\in(-\infty, 0))$$

$$\Rightarrow (s^2+2s+5)Y(s) = s+1 + 2\sin(\frac{\pi}{2})e^{-\frac{\pi}{2}s} \qquad (\because \int_{-\infty}^{\infty}\delta(t-c)f(t)\,dt = f(c))$$

$$= s+1 + 2e^{-\frac{\pi}{2}s}$$

$$\therefore \quad Y(s) = \frac{s+1}{s^2+2s+5} + 2\frac{e^{-\frac{\pi}{2}s}}{s^2+2s+5}$$

$$= \frac{s+1}{(s+1)^2+2^2} + e^{-\frac{\pi}{2}s}\frac{2}{(s+1)^2+2^2}$$

$$\Rightarrow y(t) = e^{-t}\cos(2t) + u_{\frac{\pi}{2}}(t)e^{-(t-\frac{\pi}{2})}\sin(2(t-\frac{\pi}{2}))$$

$$\boxed{y(t) = e^{-t}\cos(2t) - u_{\frac{\pi}{2}}(t)e^{-(t-\frac{\pi}{2})}\sin(2t)}$$

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b) [4] Find the values of $y(\frac{3\pi}{2})$ and $y(\frac{\pi}{4})$.

$$y(\frac{3\pi}{2}) = e^{-\frac{3\pi}{2}}\cos(3\pi) - e^{-(\frac{3\pi}{2}-\frac{\pi}{2})}\sin(3\pi) = \boxed{-e^{-\frac{3\pi}{2}}}$$

$$y(\frac{\pi}{4}) = e^{-\frac{\pi}{4}}\cos(\frac{\pi}{2}) = e^{-\frac{\pi}{4}}\times 0 = \boxed{0.}$$

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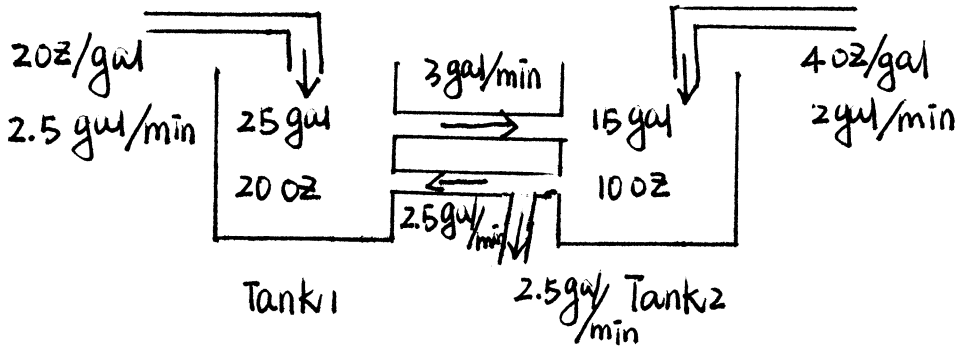
3. [25] Consider two interconnected tanks. Tank 1 initially contains 25 gal of water and 20 oz of salt, and Tank 2 initially contains 15 gal of water and 10 oz of salt. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 2.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 4 oz/gal of salt also flows into Tank 2 at a rate of 2 gal/min from outside. The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 2.5 gal/min, while the remainder leaves the system. Set up, **BUT DO NOT SOLVE**, an initial value problem which models the amount of salt in each tank at all future times.



Let $S_1(t)$ and $S_2(t)$ denote the amount (oz) of salt in Tank 1 and Tank 2, respectively

$$\frac{dS_1}{dt} = 2\left(\frac{oz}{gal}\right) \times 2.5\left|\frac{gal}{min}\right) + 2.5\left|\frac{gal}{min}\right) \times C_2(t) - 3\left|\frac{gal}{min}\right) \times C_1(t)$$

$$\frac{dS_2}{dt} = 4\left(\frac{oz}{gal}\right) \times 2\left(\frac{gal}{min}\right) + 3\left(\frac{gal}{min}\right) \times C_1(t) - 5\left|\frac{gal}{min}\right) \times C_2(t)$$

where $C_1(t)$ and $C_2(t)$ represent the concentration of mixture in tank 1 & 2.

at time t.

$$C_1(t) = \frac{S_1}{25+(2.5+2.5-3)t} = \frac{S_1}{25+2t}$$

$$C_2(t) = \frac{S_2}{15+(2+3-5)t} = \frac{S_2}{15}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \frac{dS_1}{dt} &= 5 + \frac{2.5S_2}{15} - \frac{3S_1}{25+2t} \; ; \quad S_1(0)=20 \\ \frac{dS_2}{dt} &= 8 + \frac{3S_1}{25+2t} - \frac{S_2}{3} \; ; \quad S_2(0)=10 . \end{aligned}}$$

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4. [25] Solve the initial value problem

$$x'(t) = -2x - y, \qquad x(0) = 1,$$
$$y'(t) = -7x + 4y, \qquad y(0) = 3.$$

Let $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \vec{x}' = \begin{pmatrix} -2 & -1 \\ -7 & 4 \end{pmatrix} \vec{x}$; $\vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & 1 \\ 7 & \lambda-4 \end{vmatrix} = (\lambda+2)(\lambda-4) - 7$$
$$= \lambda^2 - 2\lambda - 15 = (\lambda-5)(\lambda+3) = 0$$

$\therefore \ \lambda_1 = 5$; $\lambda_2 = -3$

For $\lambda_1 = 5$, solve $(\lambda_1 I - A)\vec{v_1} = 0$

$$\begin{pmatrix} 7 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 7v_1 + v_2 = 0$$

i.e. $v_2 = -7v_1 \Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$

For $\lambda_2 = 3$, solve $(\lambda_2 I - A)\vec{v_2} = 0$

$$\begin{pmatrix} -1 & 1 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -v_1 + v_2 = 0$$

i.e. $v_2 = v_1 \Rightarrow \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -7 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$$

At $t=0$, we have

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -7 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -7c_1 + c_2 = 3 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{1}{4} \\ c_2 = \frac{5}{4} \end{cases} \Rightarrow \boxed{\vec{x} = \frac{-1}{4}\begin{pmatrix} 1 \\ -7 \end{pmatrix} e^{5t} + \frac{5}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}}$$

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Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
7.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$f'(t)$	$sF(s) - f(0)$
9.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
10.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
11.	$u_c(t)\,f(t-c)$	$e^{-cs}F(s)$
12.	$e^{ct}f(t)$	$F(s-c)$