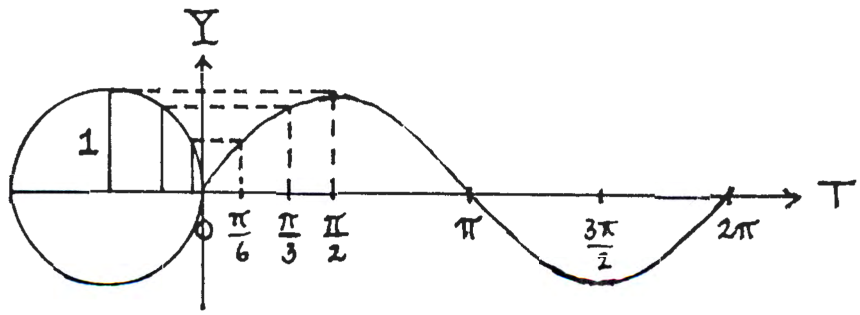
Sec. 4: Graphs of the Trig Functions

Points on the graph of $y = \sin(t)$ can be obtained using the unit circle definition of sine as indicated in the diagram.



It is apparent from the unit circle definition of sine that

$$\sin(t + 2\pi) = \sin(t)$$

for all real numbers t. Therefore the graph of $y = \sin(t)$ repeats the shape over the interval $0 \le t \le 2\pi$ indefinitely to the right and left along the t-axis.

Definition: We say that a function f is <u>periodic</u> if there exists exists a positive number p such that $f(t+p) = f(t)$ for all t in the domain of f. In this case p is called a <u>period</u> of f and the smallest such $p > 0$ is called the <u>fundamental period</u> for f.

For example, $f(t) = \sin(t)$ is periodic with fundamental period 2π.

Definition: Let f be a bounded periodic function with average value m. Then the maximum value of $|f(t) - m|$ is called the amplitude of f.

For example, $f(t) = \sin(t)$ has average value 0 and amplitude

$$\max_{0 \le t \le 2\pi} |\sin(t)| = 1.$$

All six trig functions are periodic.

function	fundamental period	amplitude
$y = \sin(t)$	2π	1
$y = \cos(t)$	2π	1
$y = \tan(t)$	π	NA
$y = \csc(t)$	2π	NA
$y = \sec(t)$	2π	NA
$y = \cot(t)$	π	NA

Please see pages 4-6 of your Trig Review notes for the graphs of the six trig functions.

Functions of the form

$$y = a\sin(bt+c) \qquad \text{or} \qquad y = a\cos(bt+c),$$

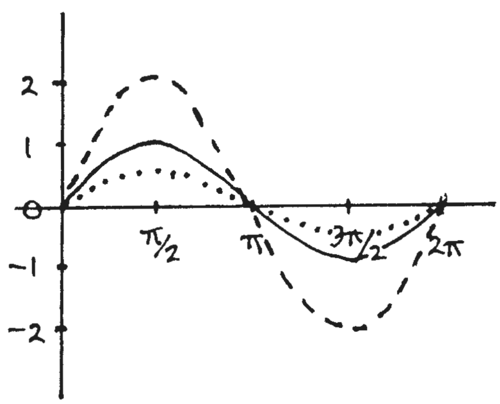
where a, b, and c are constants, are used to describe simple harmonic motion. For such functions,

$$\text{amplitude} = |a|,$$

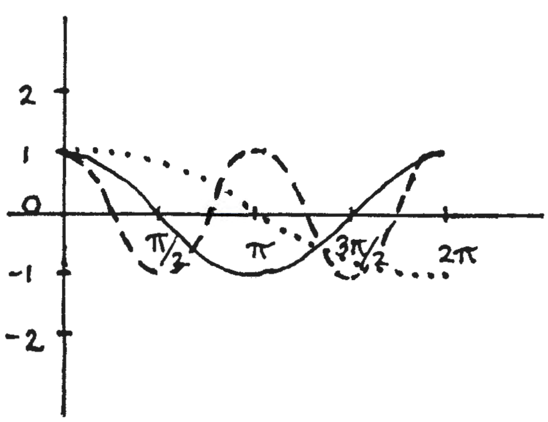
$$\text{period} = \frac{2\pi}{|b|},$$

$$\text{phase shift} = -\frac{c}{b}.$$

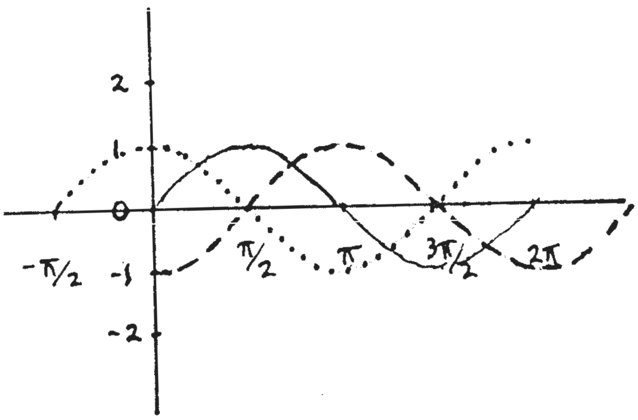
Simple examples:



	amp.	period	phs. shft
- - - $y = 2\sin(t)$	2	2π	0
—— $y = \sin(t)$	1	2π	0
····· $y = \frac{1}{2}\sin(t)$	½	2π	0



	amp.	period	phs. shft
- - - $y = \cos(2t)$	1	π	0
—— $y = \cos(t)$	1	2π	0
····· $y = \cos\left(\frac{t}{2}\right)$	1	4π	0



		amp.	period	phs. shft.
- - -	$y = \sin(t - \frac{\pi}{2})$	1	2π	$\pi/2$
——	$y = \sin(t)$	1	2π	0
· · · · ·	$y = \sin(t + \frac{\pi}{2})$	1	2π	$-\pi/2$

Example 1: Sketch the graph of $y = 3\cos(2x + \pi/2)$. Identify the amplitude, period, and phase shift.

Solution: amplitude $= |a| = |3| = 3$

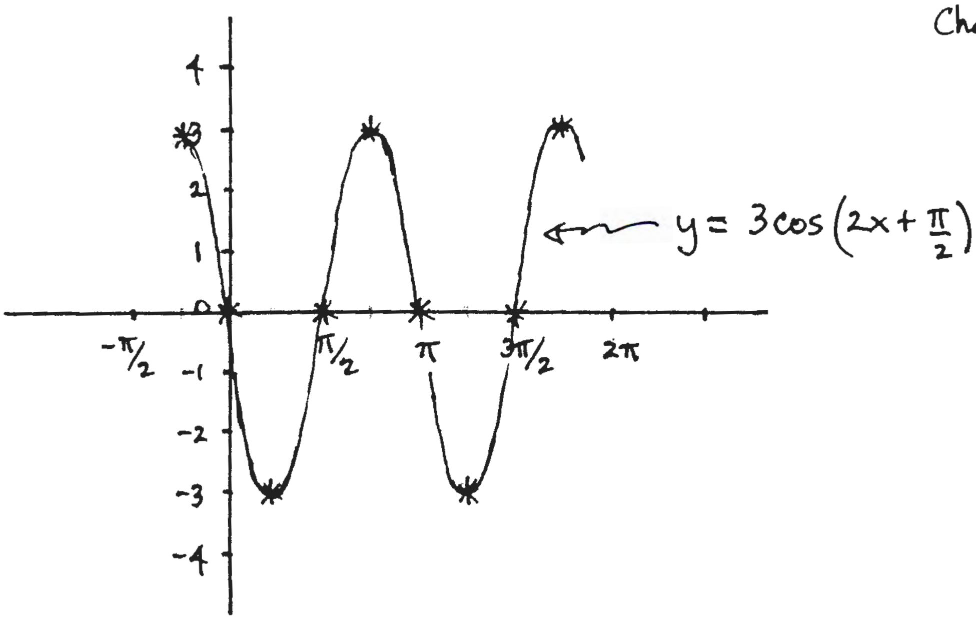
$$\text{period} = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$$

$$\text{phase shift} = -\frac{c}{b} = \frac{-\pi/2}{2} = -\frac{\pi}{4}$$

Check: $2x + \frac{\pi}{2} = 0$

$2x = -\frac{\pi}{2}$

$x = -\frac{\pi}{4}$



Any simple harmonic motion $y = A\sin(t) + B\cos(t)$ can be expressed in the equivalent form $y = C\sin(t+\theta)$ where $C = \sqrt{A^2+B^2}$ and $\tan(\theta) = \frac{B}{A}$.

Reason: Rewrite $y = A\sin(t) + B\cos(t)$ as

$$(*) \qquad y = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(t) + \frac{B}{\sqrt{A^2+B^2}}\cos(t)\right).$$

Since the point $P\left(\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right)$ lies on the unit circle $x^2+y^2=1$, there exists an angle in standard position with radian measure θ and its terminal side intersecting the unit circle at $P\left(\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right)$. By definition

$$\cos(\theta) = x = \frac{A}{\sqrt{A^2+B^2}}, \qquad \sin(\theta) = y = \frac{B}{\sqrt{A^2+B^2}},$$

and

$$\tan(\theta) = \frac{y}{x} = \frac{\frac{B}{\sqrt{A^2+B^2}}}{\frac{A}{\sqrt{A^2+B^2}}} = \frac{B}{A}.$$

Substituting in $(*)$ we have the desired expression:

$$y = \sqrt{A^2+B^2}\left(\cos(\theta)\sin(t) + \sin(\theta)\cos(t)\right) = C\sin(t+\theta).$$

Example 2: Reduce the equation $y = \sin(2x) + \cos(2x)$ to the form $y = C\sin(kx + \theta)$ and sketch the graph.

Solution: $y = 1 \cdot \sin(2x) + 1 \cdot \cos(2x)$ so by the preceding fact,

$$y = C\sin(2x + \theta)$$

where $C = \sqrt{A^2 + B^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

and $\tan(\theta) = \dfrac{B}{A} = \dfrac{1}{1} = 1$. Therefore we may take $\theta = \pi/4$. Therefore

$$y = \sin(2x) + \cos(2x) = \boxed{\sqrt{2}\sin(2x + \pi/4)}.$$

$$\text{amplitude} = \sqrt{2}$$

$$\text{period} = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$$

$$\text{phase shift} = \frac{-\pi/4}{2} = -\frac{\pi}{8}$$