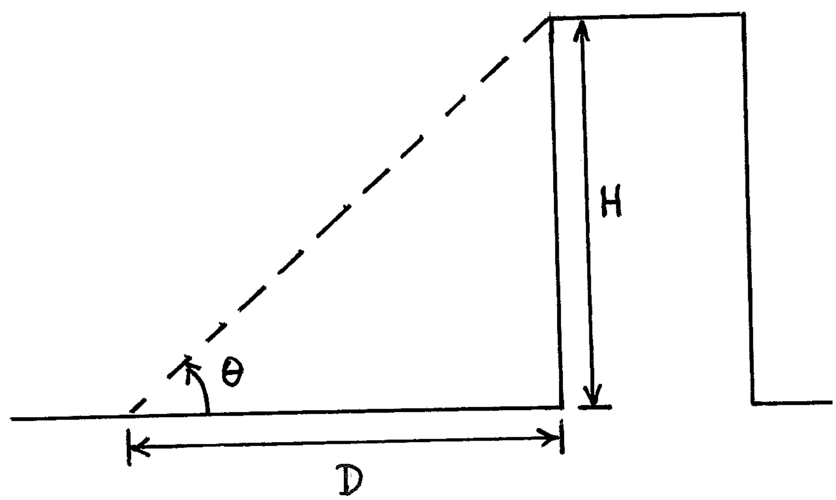
Sec. 1 : Angles and Trigonometric Functions

Trigonometry comes from two Greek words :
trigonon (triangle) and metron (measure).

Surveyors have long used triangles to indirectly measure distances.

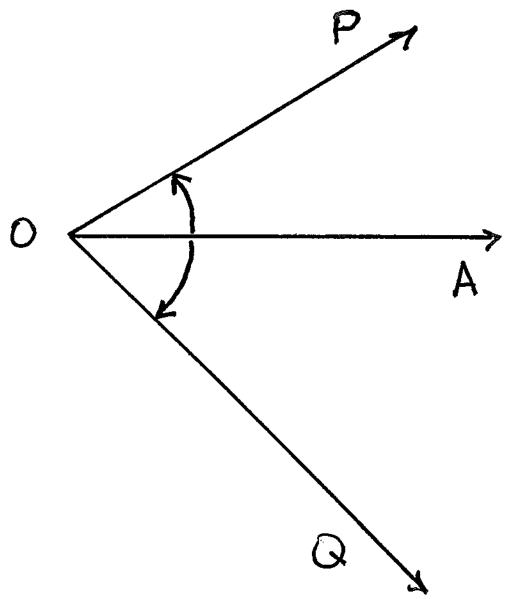


$$\tan\theta = \frac{H}{D}$$

Many modern applications of trigonometry center on the representation of functions :

$$f(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \dots$$

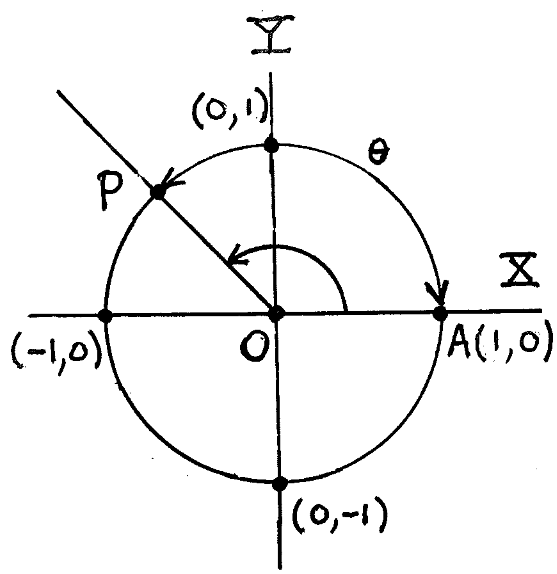
where the numbers $a_0, a_1, b_1, a_2, b_2, \dots$ are calculated from the function f – its Fourier coefficients.



The counterclockwise angle AOP has positive measure.

The clockwise angle AOQ has negative measure.

We can use a unit circle to measure angles. Recall that a circle of radius r has circumference $C = 2\pi r$. Hence a unit circle – i.e. one for which $r = 1$ – has circumference 2π.



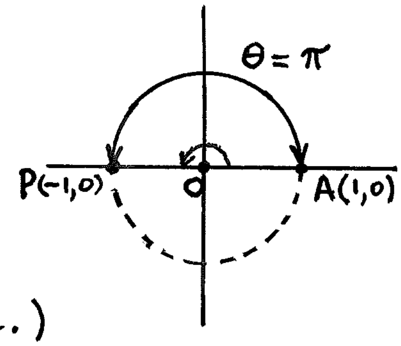
θ = radian measure of $\angle AOP$

$= \begin{cases} \text{length of arc subtended} \\ \text{by the central } \angle AOP \\ \text{in the unit circle} \end{cases}$

Example 1:

(a) A straight angle has radian measure

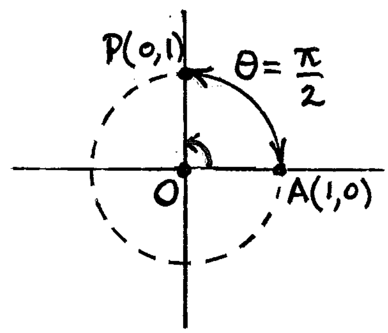
$$\theta = \frac{1}{2}(2\pi) = \boxed{\pi}.$$



(180° is the degree measure of a straight angle.)

(b) A right angle has radian measure

$$\theta = \frac{1}{4}(2\pi) = \boxed{\frac{\pi}{2}}.$$

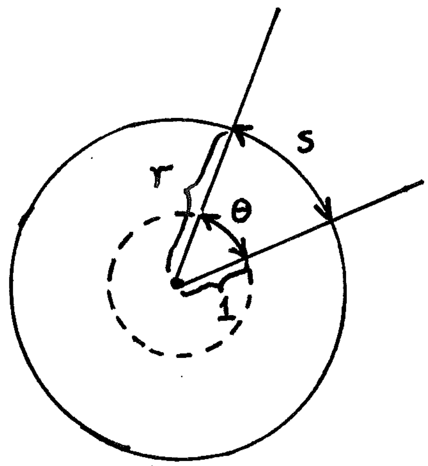


(90° is the degree measure of a right angle.)

The fact that $360^\circ = 2\pi$ radians can be used to convert degree measure to radian measure and vice versa. For example,

$$72^\circ = 72^\circ \times \frac{2\pi \text{ radians}}{360^\circ} = \boxed{\frac{2\pi}{5} \text{ radians}}$$

$$\frac{7\pi}{6} \text{ radians} = \frac{7\pi}{6} \text{ radians} \times \frac{360^\circ}{2\pi \text{ radians}} = \boxed{210^\circ}$$



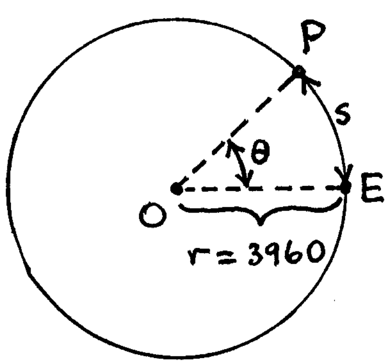
In a circle of radius r, the length of arc s subtended by a central angle of radian measure θ satisfies

$$\frac{s}{r} = \frac{\theta}{1},$$

or equivalently

$$\boxed{s = r\theta}.$$

<u>Example 2</u>: Assuming the earth to be a sphere of radius 3960 miles, find the distance of a point with latitude 42°N from the equator.



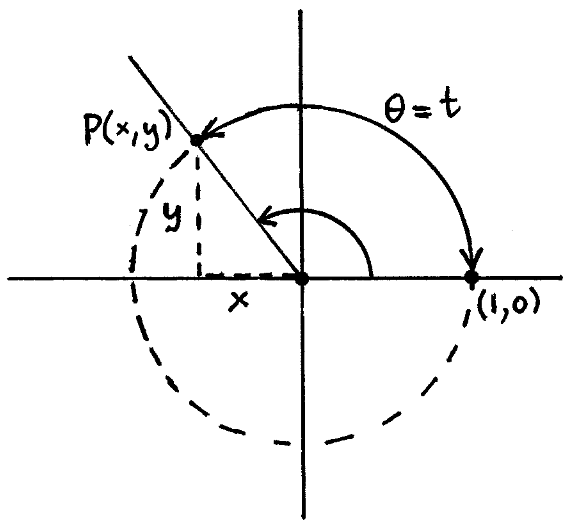
<u>Solution</u>: Since P has latitude 42°N of E, $\angle POE$ has measure 42°. Thus its radian measure is

$$\theta = 42^\circ \times \frac{2\pi \text{ radians}}{360^\circ} = \frac{7\pi}{30} \text{ radians}.$$

The distance from P to E is

$$s = r\theta = (3960)\left(\frac{7\pi}{30}\right) = 924\pi \doteq \boxed{2903 \text{ miles}}.$$

For each real number t there corresponds an angle in standard position with radian measure t. Suppose that the terminal side of this angle intersects the unit circle at the point $P(x,y)$. The six trigonometric (or circular) functions of t are defined as follows.



$$\sin(t) = y$$

$$\cos(t) = x$$

$$\tan(t) = \frac{y}{x} \quad (\text{if } x \neq 0)$$

$$\csc(t) = \frac{1}{y} \quad (\text{if } y \neq 0)$$

$$\sec(t) = \frac{1}{x} \quad (\text{if } x \neq 0)$$

$$\cot(t) = \frac{x}{y} \quad (\text{if } y \neq 0)$$

Notes: It is apparent from the definitions that

$$\tan(t) = \frac{\sin(t)}{\cos(t)}$$

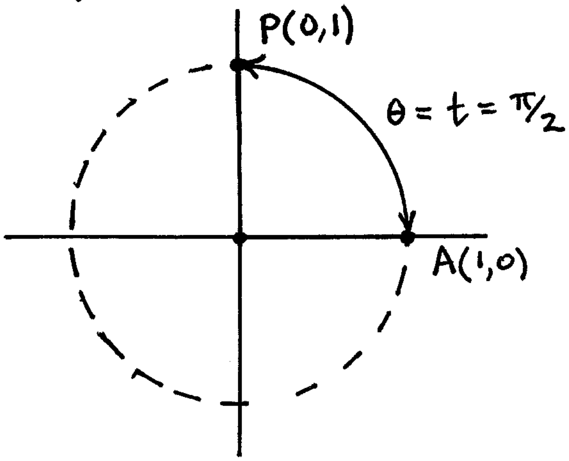
$$\csc(t) = \frac{1}{\sin(t)}$$

$$\sec(t) = \frac{1}{\cos(t)}$$

$$\cot(t) = \frac{1}{\tan(t)} = \frac{\cos(t)}{\sin(t)}.$$

Also the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$ holds because $P(x,y)$ lies on the unit circle: $x^2 + y^2 = 1$.

Example 3: Compute the values, if defined, for the six trig functions at $t = \pi/2$.



Solution: The angle in standard position with radian measure $\pi/2$ has its terminal side intersect the unit circle at the point $P(0,1)$.

Therefore

$$\sin(\pi/2) = y = \boxed{1}$$

$$\cos(\pi/2) = x = \boxed{0}$$

$$\tan(\pi/2) = \frac{y}{x} = \cancel{\frac{1}{0}} \quad \boxed{\text{undefined!}}$$

$$\csc(\pi/2) = \frac{1}{y} = \boxed{1}$$

$$\sec(\pi/2) = \frac{1}{x} = \cancel{\frac{1}{0}} \quad \boxed{\text{undefined!}}$$

$$\cot(\pi/2) = \frac{x}{y} = \frac{0}{1} = \boxed{0}$$

Example 4: If $\tan(t) = -\frac{24}{7}$ and $\sin(t) > 0$, find the values of the other five trig functions.

Solution: We want to express $\tan(t) = \frac{y}{x}$ where $P(x,y)$ lies on the unit circle $x^2 + y^2 = 1$ and $y = \sin(t) > 0$. We rewrite the given information:

$$\tan(t) = -\frac{24}{7} = -\frac{\frac{24}{\sqrt{7^2+24^2}}}{\frac{7}{\sqrt{7^2+24^2}}} = -\frac{\frac{24}{\sqrt{625}}}{\frac{7}{\sqrt{625}}} = \frac{\frac{24}{25}}{-\frac{7}{25}}.$$

The point $P(x,y) = P\left(-\frac{7}{25}, \frac{24}{25}\right)$ lies on the unit circle $x^2 + y^2 = 1$ (by the calculation above), $\frac{y}{x} = \frac{\frac{24}{25}}{\frac{-7}{25}} = -\frac{24}{7} = \tan(t)$, and $\sin(t) = y = \frac{24}{25} > 0$. Therefore

$$\boxed{\begin{array}{ll} \cos(t) = x = \frac{-7}{25} & \sec(t) = \frac{1}{\cos(t)} = -\frac{25}{7} \\ \sin(t) = y = \frac{24}{25} & \csc(t) = \frac{1}{\sin(t)} = \frac{25}{24} \\ & \cot(t) = \frac{1}{\tan(t)} = -\frac{7}{24} \end{array}}$$

Alternate Solution to Example 4: To find the point $P(x,y)$ where the terminal side of the angle t meets the unit circle, we must solve the system

$$\begin{cases} x^2 + y^2 = 1 \\ \dfrac{y}{x} = -\dfrac{24}{7}, \end{cases}$$

given that $y = \sin(t) > 0$. Substituting from the second equation into the first equation of the system gives $x^2 + \left(\frac{-24}{7}x\right)^2 = 1$, or

$$x^2 + \frac{576}{49}x^2 = 1$$

or $$49x^2 + 576x^2 = 49$$

or $$625x^2 = 49$$

or $$x = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}.$$

But $\frac{y}{x} = \tan(t) = \frac{-24}{7}$ and $y = \sin(t) > 0$, so we must select the negative sign for x: $x = -\frac{7}{25}$. Therefore $y = \frac{-24}{7}x = \left(\frac{-24}{7}\right)\left(\frac{-7}{25}\right) = \frac{24}{25}$.

Hence $P(x,y) = P\left(\frac{-7}{25}, \frac{24}{25}\right)$ and the problem is finished as before.