Mathematics 325 Exam III Summer 2000 Name: Dr. Grow

1. (40 pts.) Let $\phi(x) = x^2$ for $0 \le x \le 1$.

(a) Show that the Fourier cosine series for ϕ on $[0,1]$ is

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\pi x)}{(n\pi)^2}.$$

(b) Find the Fourier sine series for ϕ on $[0,1]$.

(c) Discuss convergence (or lack thereof) in the mean square sense for the Fourier sine and cosine series of ϕ on $[0,1]$.

(d) For which x in $[0,1]$ does the Fourier sine series of ϕ converge to $\phi(x)$? Justify your answer.

(e) ~~(d)~~ Does the Fourier cosine series of ϕ converge uniformly on $[0,1]$? Justify your answer.

(f) ~~(e)~~ Starting with the Fourier cosine series of the function $\phi(x) = x^2$ on $[0,1]$, apply Parseval's identity to help evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(a)
$$a_n = \frac{\langle \varphi, \cos(n\pi\cdot)\rangle}{\langle \cos(n\pi\cdot), \cos(n\pi\cdot)\rangle} \overset{n\ge 1}{=} 2\int_0^1 \overbrace{x^2}^{v}\overbrace{\cos(n\pi x)dx}^{dV} = \cancel{2\cdot x^2 \frac{\sin(n\pi x)}{n\pi}\Big|_0^1} - \frac{2}{n\pi}\int_0^1 \overbrace{2x}^{v}\overbrace{\sin(n\pi x)dx}^{dV}$$

$$= -\frac{4}{n\pi}\left(\frac{-x\cos(n\pi x)}{n\pi}\right)\Big|_0^1 + \cancel{\frac{4}{n\pi}\int_0^1 \frac{-\cos(n\pi x)}{n\pi}dx} = \frac{4}{(n\pi)^2}\Big[\overbrace{\cos(n\pi)}^{(-1)^n}\Big] = \frac{4(-1)^n}{(n\pi)^2}$$

$$a_0 = \frac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\therefore\ a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x) = \boxed{\frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n\cos(n\pi x)}{(n\pi)^2}}$$ Fourier cosine series of φ on $[0,1]$.

(b)
$$b_n = \frac{\langle \varphi, \sin(n\pi\cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = 2\int_0^1 \overbrace{x^2}^{v}\overbrace{\sin(n\pi x)dx}^{dV} = -\frac{2x^2\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \overbrace{2x}^{v}\overbrace{\cos(n\pi x)dx}^{dV}$$

$$= \frac{-2\overbrace{\cos(n\pi)}^{(-1)^n}}{n\pi} + \cancel{\frac{4}{n\pi}\left(\frac{x\sin(n\pi x)}{n\pi}\right)\Big|_0^1} - \frac{4}{n\pi}\int_0^1 \frac{\sin(n\pi x)}{n\pi}dx$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^2}\left(\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1 = \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^3}\Big[\overbrace{\cos(n\pi)}^{(-1)^n} - 1\Big]$$

$$\therefore \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \boxed{\sum_{n=1}^{\infty} \left\{ \frac{2(-1)^{n+1}}{n\pi} + \frac{4[(-1)^n - 1]}{(n\pi)^3} \right\} \sin(n\pi x)}$$

Fourier sine series of φ on $[0,1]$

(c) Since $\int_0^1 \varphi^2(x)\,dx = \int_0^1 x^4\,dx = \frac{1}{5} < \infty$, the Fourier cosine and Fourier sine series of φ converge to φ in the mean-square sense. (Theorem 3, p. 124)

(e) Since $\left|\frac{1}{3}\right| + \sum_{n=1}^{\infty} \left| \frac{4(-1)^n \cos(n\pi x)}{(n\pi)^2} \right| \le \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

for all $0 \le x \le 1$, the Fourier cosine series of φ is absolutely and uniformly convergent on $[0,1]$. (In fact, the Fourier cosine series of φ is uniformly convergent to φ on $[0,1]$.)

(f) $$\sum_{n=0}^{\infty} a_n^2 \int_0^1 \cos^2(n\pi x)\,dx = \int_0^1 \varphi^2(x)\,dx \qquad \text{(Parseval)}$$

$$\Rightarrow \left(\frac{1}{3}\right)^2 + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{(n\pi)^2}\right)^2 \cdot \frac{1}{2} = \frac{1}{5} \qquad \text{(by (a) and (c))}$$

$$\frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{8} \cdot \frac{4}{45} = \boxed{\frac{\pi^4}{90}}$$

(d) Since $\varphi(x) = x^2$ and $\varphi'(x) = 2x$ are continuous on $[0,1]$, the Fourier sine series of φ at x converges to $\varphi(x)$ for all $0 < x < 1$. (Theorem 4, p. 125)

At the endpoints: The Fourier sine series of φ at 0 is (clearly) 0 and this is equal to $\varphi(0) = 0^2 = 0$. The Fourier sine series of φ at 1 is (clearly) 0 and this not equal to $\varphi(1) = 1^2 = 1$.

Conclusion: $\boxed{\text{The Fourier sine series of } \varphi \text{ at } x \text{ converges to } \varphi(x) \text{ for all } 0 \le x < 1}$.

2. (30 pts.) (a) Show that the operator T defined on

$$V = \{ f \in C^2(0,1] : f(1) = 0,\ f \text{ and } f' \text{ bounded on } (0,1] \}$$

$$Tf(x) = \frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right) - \frac{n^2}{x^2} f(x) \qquad (0 < x \le 1)$$

is hermitian. (Please use the inner product on V given by

$$\langle f,g \rangle = \int_0^1 f(x)\overline{g(x)}\,x\,dx \ .)$$

(b) Are the eigenvalues of T on V real? Why or why not?
(c) Are the eigenvalues of T on V nonnegative? Why or why not?
(d) Are the eigenfunctions of T on V, corresponding to distinct eigenvalues, orthogonal on (0,1)? Why or why not?

Let $f, g \in V$.

(a) $$\langle Tf, g\rangle = \int_0^1 Tf(x)\overline{g(x)}\,x\,dx = \int_0^1 \cancel{\tfrac{1}{x}}\, \overbrace{\frac{d}{dx}\left(x\frac{df}{dx}\right)}^{dV} \overbrace{\overline{g(x)}}^{v}\,\cancel{x}\,dx - n^2\int_0^1 \frac{1}{x^2} f(x)\overline{g(x)}\,x\,dx$$

$$= \underbrace{x\frac{df}{dx}\overline{g(x)}\Big|_0^1}_{} - \int_0^1 x\frac{df}{dx}\,\frac{d\overline{g}}{dx}\,dx - n^2\int_0^1 \frac{1}{x^2} f(x)\overline{g(x)}\,x\,dx$$

0 because $f'(1)\overline{g(1)} = 0$ and $\lim_{\varepsilon\to 0^+} \varepsilon f'(\varepsilon)\overline{g(\varepsilon)} = 0$

$$= \underbrace{-x\frac{d\overline{g}}{dx} f(x)\Big|_0^1}_{} + \int_0^1 f(x)\frac{d}{dx}\left(x\frac{d\overline{g}}{dx}\right)dx - n^2\int_0^1 \frac{1}{x^2} f(x)\overline{g(x)}\,x\,dx$$

0 because $-\overline{g'(1)}f(1) = 0$ and $\lim_{\varepsilon\to 0^+} -\varepsilon\,\overline{g'(\varepsilon)}f(\varepsilon) = 0$

$$= \int_0^1 f(x)\overline{\left[\frac{1}{x}\frac{d}{dx}\left(x\frac{dg}{dx}\right) - \frac{n^2}{x^2}g(x)\right]}\,x\,dx = \langle f, Tg\rangle$$ (Thus, T is hermitian on V.)

(b) Yes, the eigenvalues of T on V are real by the argument given in class.

(d) Yes, the eigenfunctions of T on V, corresponding to distinct eigenvalues, are orthogonal on (0,1) (relative to the weight function x) by the argument given in class.

(c) No, for let λ be an eigenvalue of T on V and let f be a corresponding eigenfunction. Taking $g = f$ in the first two lines of (a),

$$\lambda\langle f,f\rangle = \langle \lambda f, f\rangle = \langle Tf, f\rangle = -\int_0^1 x\left|\frac{df}{dx}\right|^2 dx - n^2\int_0^1 |f(x)|^2\frac{1}{x}\,dx < 0.$$ Thus $\boxed{\lambda < 0}$

3. (30 pts.) Find a solution to

$$u_{tt} = u_{xx} \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

subject to the homogeneous boundary/initial conditions

$$u_x(0,t) = 0 = u_x(1,t) \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \le t,\ 0 \le x \le 1,$$

and the nonhomogeneous initial condition

$$u(x,0) = x^2 \quad \text{for } 0 \le x \le 1.$$

Is your solution the only possible one? Give a complete justification of your answer.

$u(x,t) = X(x)T(t)$ (nontrivial solns. to the homogeneous part of the problem)

$$X(x)T''(t) = X''(x)T(t) \quad \Rightarrow \quad -\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda .$$

$$\left.\begin{array}{l} u_x(0,t) = X'(0)T(t) = 0 \\ u_x(1,t) = X'(1)T(t) = 0 \end{array}\right\} \ (0 \le t < \infty) \text{ and } u = XT \text{ nontrivial} \Rightarrow X'(0) = 0 = X'(1)$$

$$u_t(x,0) = X(x)T'(0) = 0 \quad (0 \le x \le 1) \quad \Rightarrow T'(0) = 0$$

$$X''(x) + \lambda X(x) = 0 \quad (0 < x < 1),\ X'(0) = X'(1) = 0$$

$$T''(t) + \lambda T(t) = 0 \quad (0 < t < \infty),\ T'(0) = 0$$

Eigenvalues: $\lambda_n = (n\pi)^2 \quad (n = 0, 1, 2, \ldots)$

Eigenfunctions: $X_n(x) = \cos(n\pi x) \quad (n = 0, 1, 2, \ldots)$

Solution to t-equation corresponding to $\lambda = \lambda_n$: $T_n(t) = \cos(n\pi t) \quad (n = 0, 1, 2, \ldots)$

$\therefore\ u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\cos(n\pi t)$ is a formal solution to the homogeneous part of the problem. We want to choose the constants $a_0, a_1, \ldots$ so the non-homogeneous I.C. is satisfied:

$$x^2 \overset{\text{want}}{=} u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{for } 0 \le x \le 1.$$

By problem 1, $a_0 = \frac{1}{3}$ and $a_n = \frac{4(-1)^n}{(n\pi)^2}$ $(n \ge 1)$ suffice. Thus

$$\boxed{u(x,t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\pi x)\cos(n\pi t)}{(n\pi)^2}}$$

solves the problem. This is the only solution, too. To see this, suppose that $u = u_2(x,t)$ were another solution. Then

$$w(x,t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\pi x)\cos(n\pi t)}{(n\pi)^2} - u_2(x,t)$$

solves

$$w_{tt} = w_{xx} \qquad (0<x<1,\ 0<t<\infty),$$

$$w_x(0,t) = 0 = w_x(1,t) \qquad (0 \le t < \infty),$$

$$w(x,0) = 0 = w_t(x,0) \qquad (0 \le x \le 1).$$

Consider the energy function of w,

$$E(t) = \int_0^1 \left[w_t^2(x,t) + w_x^2(x,t) \right] dx \qquad (t \ge 0).$$

Observe that

$$E'(t) = \int_0^1 \frac{\partial}{\partial t}\left[w_t^2(x,t) + w_x^2(x,t) \right] dx$$

$$= 2\int_0^1 w_t(x,t) w_{tt}(x,t)\,dx + 2\int_0^1 w_x(x,t) w_{xt}(x,t)\,dx$$

$$= 2\int_0^1 \overbrace{w_t(x,t)}^{v}\, \overbrace{w_{xx}(x,t)\,dx}^{dv} + 2\int_0^1 w_x(x,t) w_{xt}(x,t)\,dx$$

$$= \underbrace{2 w_t(x,t) w_x(x,t)\Big|_0^1}_{0 \text{ since } w_x(1,t) = 0 = w_x(0,t)} \underbrace{- 2\int_0^1 w_{tx}(x,t) w_x(x,t)\,dx + 2\int_0^1 w_x(x,t) w_{xt}(x,t)\,dx}_{0 \text{ since } w_{tx} = w_{xt}}$$

$$= 0$$

Thus $E(t) = \text{constant}$ for all $t \ge 0$. But

$$E(0) = \int_0^1 \Big[\underbrace{w_t^2(x,0)}_{0 \text{ since } w_t(x,0)=0 \text{ for } 0 \le x \le 1} + \overbrace{w_x^2(x,0)}^{0 \text{ since } w_x(x,0) = \lim_{h\to 0} \frac{\overbrace{w(x+h,0)}^{0} - \overbrace{w(x,0)}^{0}}{h} = 0 \text{ for } x \text{ in } [0,1]} \Big] dx = 0,$$

so $E(t) = 0$ for all $t \ge 0$. By continuity and positivity of the integrand of

E, it follows that $w_t^2(x,t) + w_x^2(x,t) = 0$ for all $0 \le x \le 1$ and $t \ge 0$. Consequently $0 = w_t(x,t) = w_x(x,t)$ for all $0 \le x \le 1$ and $t \ge 0$, from which it follows that $w(x,t) = \text{constant}$ on $0 \le x \le 1$, $0 \le t < \infty$. But then $w(x,0) = 0$ for $0 \le x \le 1$ implies $w(x,t) = 0$ for all $0 \le x \le 1$ and $0 \le t < \infty$. That is,

$$u_2(x,t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\pi x)\cos(n\pi t)}{(n\pi)^2} \qquad (0 \le x \le 1,\ 0 \le t < \infty).$$