Mathematics 325 Final Exam Fall 2000 Name: Dr. Grow

All problems are of equal value. I choose to have my test be worth (circle one) 200 points / 300 points.

1. (a) Find the general solution in the xy-plane of

(*) $$yu_x - xu_y = 0.$$

(b) Sketch and name some characteristic curves of (*).

2. (a) Classify the partial differential equation

(+) $$u_{xx} - 4u_{xt} + 4u_{tt} = 0$$

as elliptic, parabolic, or hyperbolic.

(b) If it is possible, derive the general solution to (+) in the xt-plane.

(c) Find the solution to (+) in the upper xt-halfplane which satisfies the initial conditions

$$u(x,0) = xe^{2x} + 4x^2 \quad \text{and} \quad u_t(x,0) = (x-2)e^{2x} + 4x \quad \text{for } -\infty < x < \infty.$$

3. Let ϕ be an absolutely integrable function on $(-\infty,\infty)$. Use Fourier transform methods to find a solution to

$$u_t - u_{xx} + 3u = 0 \qquad \text{in } -\infty < x < \infty,\ 0 < t < \infty,$$

which satisfies the initial condition

$$u(x,0) = \phi(x) \qquad \text{for } -\infty < x < \infty.$$

4. Consider an infinite string with linear density $\rho = 1$ and tension $T = 1$, initially occupying the position of the x-axis. At time $t = 0$ and at general horizontal position x, the string is displaced vertically by an amount e^{-x^2} and released with velocity $2xe^{-x^2}$.

(a) Find the vertical displacement of the string as a function of position x and time t, and simplify your formula as much as possible.

(b) Sketch profiles of the vertical displacement function at times $t = 1$, $t = 2$, and $t = 3$.

5. The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Centigrade on its inner boundary. On its outer boundary, the temperature distribution of the material staisfies $u_r = -\gamma$ where γ is a positive constant.

(a) Find the temperature distribution function for the material.

(b) What are the hottest and coldest temperatures in the material?

(c) Is it possible to choose γ so that the temperature on the outer boundary is 20 degrees Centigrade? Support your answer.

6. (a) Find a solution to

$$\nabla^2 u = 0 \quad \text{in the cube } C\text{: } 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,$$

subject to the boundary conditions

$$u(x,y,1) = \sin(\pi x)\sin^3(\pi y) \qquad \text{for } 0 \le x \le 1,\ 0 \le y \le 1,$$

and $u = 0$ on the other five faces of the cube C.

(b) State the maximum/minimum principle for harmonic functions, and use it to show that the problem in part (a) has only one solution.

7. (a) Find a solution to

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \qquad \text{in } 0 < x < \pi,\ 0 < t < \infty,$$

satisfying

(2) $$u_x(0,t) = 0 = u_x(\pi,t) \qquad \text{for } t \ge 0,$$

(3) $$u(x,0) = 0 \qquad \text{for } 0 \le x \le \pi,$$

(4) $$u_t(x,0) = x^2 \qquad \text{for } 0 \le x \le \pi.$$

(b) If $u = u(x,t)$ satisfies (1)-(2), show that its energy

$$E(t) = \frac{1}{2}\int_0^\pi [u_t^2(x,t) + u_x^2(x,t)]dx$$

is decreasing on $0 \le t < \infty$.

(c) Is there only one solution to the problem in part (a)? Why or why not?

#1. $y u_x - x u_y = 0$ $(*)$

$D_{[y;-x]} u = 0$

Characteristic curves: $\frac{dy}{dx} = \frac{-x}{y} \Rightarrow y\,dy = -x\,dx$

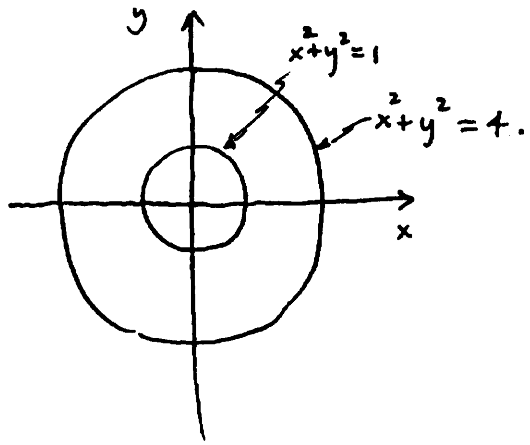
$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$

$\Rightarrow \boxed{x^2+y^2=k \quad \text{(circles with center at (0,0))}}$.

Along such curves,

$u(x,y) = u(x, \pm\sqrt{k-x^2}) = u(0, \pm\sqrt{k}) = f(k)$

$\therefore$ $\boxed{u(x,y) = f(x^2+y^2)}$ is the general solution to $(*)$, where $f = f(t)$ is a C^1-function on $[0,\infty)$.



#2. (+) $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ $\qquad B^2 - 4AC = 16 - 4(1)(4) = 0$

(a) $\boxed{\text{parabolic}}$

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)^2 u = 0$$

Let $\begin{cases} \xi = 2x + t \\ \eta = -x + 2t \end{cases}$

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 2\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta} \\ \frac{\partial v}{\partial t} &= \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial v}{\partial \xi} + 2\frac{\partial v}{\partial \eta} \end{aligned}\right\} \text{ I.e. } \quad \begin{aligned} \frac{\partial}{\partial x} &= 2\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta} \end{aligned}$$

$$\therefore \quad \frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} = \cancel{2\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta} - 2\left(\cancel{\frac{\partial}{\partial \xi}} + 2\frac{\partial}{\partial \eta}\right)$$
$$= -5\frac{\partial}{\partial \eta}$$

(+) is equivalent to $\left(-5\frac{\partial}{\partial \eta}\right)^2 u = 0$

$$\cancel{25}\,\frac{\partial^2 u}{\partial \eta^2} = 0$$

$$\therefore \quad u = a(\xi)\eta + b(\xi)$$

(b) General Solution: $\boxed{u(x,t) = f(2x+t)(-x+2t) + g(2x+t)}$

where f and g are C^2-functions of a single real variable.

$$u_t(x,t) = f'(2x+t)(-x+2t) + 2f(2x+t) + g'(2x+t)$$

$$\begin{cases} xe^{2x} + 4x^2 = u(x,0) = -xf(2x) + g(2x) \\ (x-2)e^{2x} + 4x = u_t(x,0) = -xf'(2x) + 2f(2x) + g'(2x) \end{cases}$$ ← (Differentiate first equation.)

$$\Rightarrow \begin{cases} (2x+1)e^{2x} + 8x = -2xf'(2x) - f(2x) + 2g'(2x) \\ (x-2)e^{2x} + 4x = -xf'(2x) + 2f(2x) + g'(2x) \end{cases}$$ ← (multiply second equation by 2 and subtract from first.)

#2 (cont.) $\quad 5e^{2x} = -5f(2x) \Rightarrow f(v) = -e^{v}.$

Substituting for f in $xe^{2x} + 4x^2 = -xf(2x) + g(2x)$ gives

$$xe^{2x} + 4x^2 = xe^{2x} + g(2x)$$

$$\Rightarrow (2x)^2 = g(2x) \quad \Rightarrow \quad g(v) = v^2.$$

$$\therefore \quad u(x,t) = f(2x+t)(-x+2t) + g(2x+t)$$

$$= \boxed{e^{2x+t}(-2t+x) + (2x+t)^2} \qquad (c)$$

#3 $\mathcal{F}(u_t - u_{xx} + 3u)(\xi) = \mathcal{F}(0)(\xi)$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 3\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+3)\mathcal{F}(u)(\xi) = 0$$

Integrating factor: $\mu(t) = e^{\int(\xi^2+3)\,dt} = e^{(\xi^2+3)t}$

$$e^{(\xi^2+3)t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+3)e^{(\xi^2+3)t}\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\left[e^{(\xi^2+3)t}\mathcal{F}(u)(\xi)\right] = 0$$

$$e^{(\xi^2+3)t}\mathcal{F}(u)(\xi) = c(\xi)$$

$$\mathcal{F}(u)(\xi) = c(\xi)e^{-(\xi^2+3)t}$$

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)e^{-0} = c(\xi)$$

$$\therefore\ \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t}\cdot e^{-3t}$$

By Table entry I, $\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}$. Taking $\frac{1}{4a} = t$

(i.e. $a = \frac{1}{4t}$), we have $\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi) = e^{-\xi^2 t}$.

$$\therefore\ \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)\cdot e^{-3t}.$$

Using the convolution fact, $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ gives

#3 (cont.) $\mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{1}{\sqrt{2t}} e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)\cdot e^{-3t}$

$$= \mathcal{F}\left(e^{-3t}\cdot \varphi * \frac{1}{\sqrt{4\pi t}} e^{-\frac{(\cdot)^2}{4t}}\right)(\xi).$$

By the Fourier inversion theorem,

$$u(x,t) = e^{-3t}\cdot\left(\varphi * \frac{1}{\sqrt{4\pi t}} e^{-\frac{(\cdot)^2}{4t}}\right)(x)$$

$$= \boxed{e^{-3t}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\,\varphi(y)\, e^{-\frac{(x-y)^2}{4t}}\,dy}$$

for all $-\infty < x < \infty$ and all $0 < t < \infty$.

#4 $\quad \rho u_{tt} - T u_{xx} = 0 \quad , \quad u(x,0) = e^{-x^2} \quad , \quad u_t(x,0) = 2x e^{-x^2}$

$u_{tt} - u_{xx} = 0 \quad , \quad$ "

$$u(x,t) = \frac{1}{2}\left[\varphi(x-t) + \varphi(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\,d\xi \qquad (d'Alembert)$$

where $\varphi(x) = e^{-x^2}$ and $\psi(x) = 2x e^{-x^2}$.

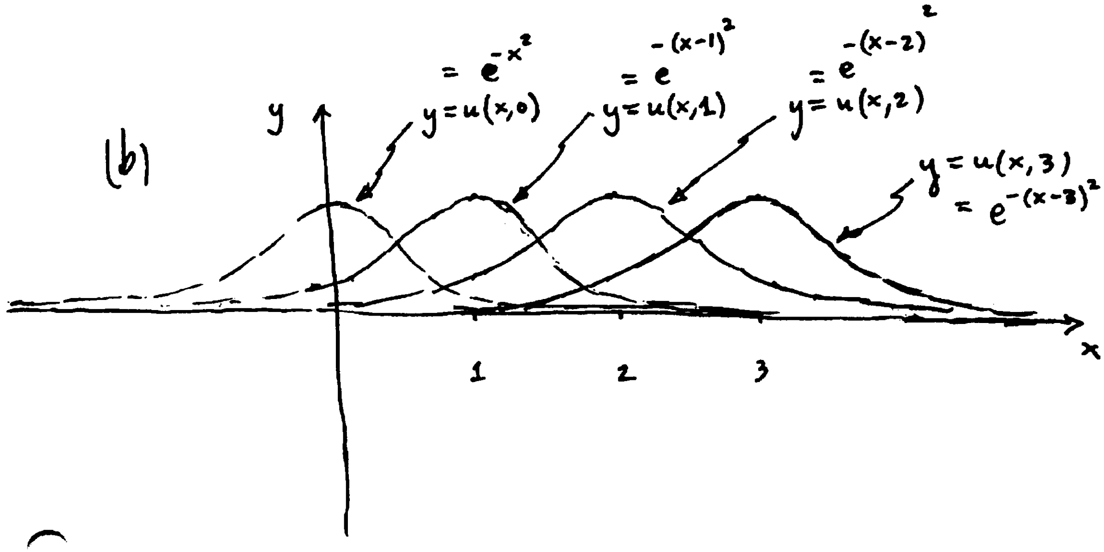
$$\therefore \; u(x,t) = \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} 2\xi e^{-\xi^2}\,d\xi$$

$$= \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\left(-e^{-\xi^2}\right)\Big|_{\xi=x-t}^{\xi=x+t}$$

$$= \frac{1}{2}\left[e^{-(x-t)^2} + \cancel{e^{-(x+t)^2}}\right] - \frac{1}{2}\left[\cancel{e^{-(x+t)^2}} - e^{-(x-t)^2}\right]$$

(a) $\boxed{u(x,t) = e^{-(x-t)^2}}$

(b)



#5

$\nabla^2 u = 0 \quad \text{in} \quad 1 < r < 2,$

$u(1,\theta,\varphi) = 100 \quad \text{for all } -\pi \le \theta < \pi, \ 0 \le \varphi \le \pi,$

$u_r(2,\theta,\varphi) = -\gamma \qquad$ " $\qquad$ " .

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\varphi\varphi} + \cot(\varphi)u_\varphi + \frac{1}{\sin^2(\varphi)}u_{\theta\theta}\right] = 0$$

Assume u is a radial function (i.e. $u = u(r)$ independent of θ and φ).

$$u_{rr} + \frac{2}{r}u_r = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

$$\Rightarrow r^2 u_r = c_1$$

$$u_r = \frac{c_1}{r^2}$$

$$\Rightarrow \quad u = -\frac{c_1}{r} + c_2 .$$

$$-\gamma = u_r(2) = \frac{c_1}{(2)^2} \Rightarrow c_1 = -4\gamma.$$

$$100 = u(1) = -\frac{(-4\gamma)}{1} + c_2 \Rightarrow c_2 = 100 - 4\gamma$$

(a) $\boxed{u(r) = \frac{4\gamma}{r} + 100 - 4\gamma}$

(b) hottest temperature: $u(1) = 4\gamma + 100 - 4\gamma = \boxed{100}$ degrees C.

coldest temperature: $u(2) = \frac{4\gamma}{2} + 100 - 4\gamma = \boxed{100 - 2\gamma}$ degrees C.

(c) Yes, if $\boxed{\gamma = 40}$ then

$u(2) = 100 - 2(40) = 20$ deg. C.

#6 $u_{xx} + u_{yy} + u_{zz} = 0$ in $0<x<1, 0<y<1, 0<z<1$,

$u(0,y,z) = u(1,y,z) = 0$ for all $0 \le y \le 1, 0 \le z \le 1$

$u(x,0,z) = u(x,1,z) = 0$ " " $0 \le x \le 1, 0 \le z \le 1$

$u(x,y,0) = 0$ and $u(x,y,1) = \sin(\pi x)\sin^3(\pi y)$ for $0 \le x \le 1, 0 \le y \le 1$.

$u(x,y,z) = X(x)Y(y)Z(z)$ (nontrivial solution to homogeneous part of the problem)

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \lambda$$

$$\Rightarrow \quad \frac{-Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda = \mu$$

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(1) = 0$$

$$Y''(y) + \mu Y(y) = 0, \quad Y(0) = Y(1) = 0$$

$$Z''(z) - (\lambda+\mu)Z(z) = 0, \quad Z(0) = 0$$

$$\Rightarrow \quad X_\ell(x) = \sin(\ell\pi x), \quad \lambda_\ell = (\ell\pi)^2 \qquad (\ell = 1,2,3,\dots)$$

$$Y_m(y) = \sin(m\pi y), \quad \mu_m = (m\pi)^2 \qquad (m = 1,2,3,\dots)$$

$$Z_{\ell,m}(z) = \sinh(\pi z\sqrt{\ell^2+m^2})$$

$$u(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell,m} \sin(\ell\pi x)\sin(m\pi y)\sinh(\pi z\sqrt{\ell^2+m^2})$$

#6 (cont.) $\sin(\pi x)\sin^3(\pi y) = u(x,y,1) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} c_{l,m} \sinh(\pi\sqrt{l^2+m^2})\sin(l\pi x)\sin(m\pi y)$

$$\sin(\pi x)\left[\frac{3}{4}\sin(\pi y) - \frac{1}{4}\sin(3\pi y)\right] = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} c_{l,m}\sinh(\pi\sqrt{l^2+m^2})\sin(l\pi x)\sin(m\pi y)$$

$c_{1,1}\sinh(\pi\sqrt{2}) = \frac{3}{4}$, $c_{1,3}\sinh(\pi\sqrt{10}) = -\frac{1}{4}$, and all other $c_{l,m} = 0$.

(a) $$\boxed{u(x,y,z) = \frac{3\sin(\pi x)\sin(\pi y)\sinh(\pi z\sqrt{2})}{4\sinh(\pi\sqrt{2})} - \frac{\sin(\pi x)\sin(3\pi y)\sinh(\pi z\sqrt{10})}{4\sinh(\pi\sqrt{10})}}$$

(b) (Weak) Max/min Principle for harmonic functions:
If $\nabla^2 u = 0$ in an open bounded region Ω and u is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ then $\max_{\vec{x}\in\overline{\Omega}} u(\vec{x}) = \max_{\vec{x}\in\partial\Omega} u(\vec{x})$ and $\min_{\vec{x}\in\Omega} u(\vec{x}) = \min_{\vec{x}\in\partial\Omega} u(\vec{x})$.

Suppose $u = v(x,y,z)$ were another solution to the problem in part (a). Then $w = u(x,y,z) - v(x,y,z)$ solves $\nabla^2 w = 0$ in C, w is continuous in $\overline{C} = C \cup \partial C$, and $w = 0$ on ∂C. Hence

$$0 = \min_{(x,y,z)\in\partial C} w(x,y,z) \le w(x_0,y_0,z_0) \le \max_{(x,y,z)\in\partial C} w(x,y,z) = 0$$

for all $(x_0,y_0,z_0) \in \overline{C}$. That is, $0 = w(x_0,y_0,z_0) = u(x_0,y_0,z_0) - v(x_0,y_0,z_0)$ for all $(x_0,y_0,z_0) \in \overline{C}$, and hence $u = v$.

#7 $\quad u(x,t) = X(x)T(t) \longrightarrow \begin{cases} u_{tt} - u_{xx} + 2u_t = 0 \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = 0 \end{cases}$

$$X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$$

$$\Rightarrow \frac{T''(t) + 2T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0, & X'(0) = X'(\pi) = 0 \\ T''(t) + 2T'(t) + \lambda T(t) = 0, & T(0) = 0 \end{cases}$$

$$\lambda_n = n^2, \quad X_n(x) = \cos(nx) \qquad (n = 0, 1, 2, \ldots)$$

$$\Rightarrow T_n''(t) + 2T_n'(t) + n^2 T_n(t) = 0, \quad T_n(0) = 0 .$$

$T_n(t) = e^{\alpha_n t}$ leads to $\alpha_n^2 + 2\alpha_n + n^2 = 0 \Rightarrow \alpha_n = \dfrac{-2 \pm \sqrt{4 - 4n^2}}{2}$

$$\therefore \alpha_0 = -1 \pm 1 = \begin{cases} 0 \\ -2 \end{cases}, \quad \alpha_1 = -1, \quad \alpha_n = -1 \pm i\sqrt{n^2-1} \quad \text{for } n \geq 2 .$$

$$\therefore T_0(t) = a_0 + b_0 e^{-2t}, \quad T_1(t) = a_1 e^{-t} + b_1 t e^{-t}, \quad \underset{(n \geq 2)}{T_n(t)} = e^{-t}\left[a_n \cos(t\sqrt{n^2-1}) + b_n \sin(t\sqrt{n^2-1})\right]$$

$T_n(0) = 0 \Rightarrow a_0 + b_0 = 0$ and $a_n = 0$ $(n \geq 1)$. Up to a constant factor,

$$\therefore \quad u_0(x,t) = 1 - e^{-2t}, \quad u_1(x,t) = te^{-t}\cos(x), \quad \underset{(n \geq 2)}{u_n(x,t)} = e^{-t}\sin(t\sqrt{n^2-1})\cos(nx) .$$

$$\therefore u(x,t) = b_0(1 - e^{-2t}) + b_1 t e^{-t}\cos(x) + \sum_{n=2}^{\infty} b_n e^{-t} \sin(t\sqrt{n^2-1})\cos(nx)$$

$$u_t(x,t) = 2b_0 e^{-2t} + b_1(1-t)e^{-t}\cos(x) + \sum_{n=2}^{\infty} b_n e^{-t}\left[-\sin(t\sqrt{n^2-1}) + \sqrt{n^2-1}\cos(t\sqrt{n^2-1})\right]\cos(nx)$$

#7 (cont.) $x^2 = u_t(x,0) = 2b_0 + b_1\cos(x) + \sum_{n=2}^{\infty} b_n\sqrt{n^2-1}\cos(nx)$ for all $0 \le x \le \pi$.

$$\therefore\ 2b_0 = \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^\pi x^2\,dx}{\int_0^\pi 1^2\,dx} = \frac{\pi^2}{3},$$

$$b_1 = \frac{\langle x^2, \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}, \text{ and } \sqrt{n^2-1}\,b_n = \frac{\langle x^2, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle}\quad (n \ge 2).$$

$$\frac{\langle x^2, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{\int_0^\pi x^2\cos(nx)\,dx}{\int_0^\pi \cos^2(nx)\,dx} = \frac{2}{\pi}\int_0^\pi \overbrace{x^2}^{u}\overbrace{\cos(nx)\,dx}^{dv} = \cancel{\frac{2}{\pi}\left(x^2\frac{\sin(nx)}{n}\right)\Bigg|_0^\pi}^{\,0} - \frac{2}{\pi}\int_0^\pi 2x\frac{\sin(nx)}{n}\,dx$$

$$= \frac{-4}{\pi n}\int_0^\pi \overbrace{x}^{u}\overbrace{\sin(nx)\,dx}^{dv} = \frac{-4}{\pi n}\left(-\frac{x\cos(nx)}{n}\right)\Bigg|_0^\pi + \cancel{\frac{4}{\pi n}\int_0^\pi \frac{\cos(nx)}{n}\,dx}^{\,0} = \frac{4(-1)^n}{n^2}.$$

$$\therefore\ b_0 = \frac{\pi^2}{6},\quad b_1 = -4,\ \text{and}\ b_n = \frac{4(-1)^n}{n^2\sqrt{n^2-1}}\quad (n \ge 2).$$

(a) $$\boxed{u(x,t) = \frac{\pi^2}{6}(1-e^{-2t}) - 4te^{-t}\cos(x) + \sum_{n=2}^{\infty}\frac{4(-1)^n e^{-t}\sin(t\sqrt{n^2-1})\cos(nx)}{n^2\sqrt{n^2-1}}}.$$

(b) $$E'(t) = \frac{d}{dt}\left\{\frac{1}{2}\int_0^\pi\left[u_t^2(x,t) + u_x^2(x,t)\right]dx\right\} = \frac{1}{2}\int_0^\pi \frac{\partial}{\partial t}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$$

$$= \int_0^\pi\left[u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t)\right]dx \overset{\text{by (1)}}{=} \int_0^\pi u_t(x,t)\left[u_{xx}(x,t) - 2u_t(x,t)\right]dx + \int_0^\pi \overbrace{u_x(x,t)}^{u}\overbrace{u_{xt}(x,t)\,dx}^{dv}$$

$$= \cancel{\int_0^\pi u_t(x,t)u_{xx}(x,t)\,dx} - 2\int_0^\pi u_t^2(x,t)\,dx + \cancel{u_x(x,t)u_t(x,t)\Big|_0^\pi}^{\,0\ (\text{by }(2))} - \cancel{\int_0^\pi u_t(x,t)u_{xx}(x,t)\,dx}$$

$$= -2\int_0^\pi u_t^2(x,t)\,dx \le 0.$$ $\boxed{\text{Thus } E \searrow \text{ on } 0 \le t < \infty.}$

#7 (cont.) (c) $\boxed{\text{There is only one solution to (1)-(2)-(3)-(4).}}$ To see this, let $u = u(x,t)$ denote the solution found above in part (a) and let $u = u_1(x,t)$ be another solution to (1)-(2)-(3)-(4). Then $v(x,t) = u(x,t) - u_1(x,t)$ solves

(1') $\quad v_{tt} - v_{xx} + 2v_t = 0 \quad$ in $0 < x < \pi,\ 0 < t < \infty,$

(2') $\quad v_x(0,t) = 0 = v_x(\pi,t) \quad$ for $t \geq 0,$

(3') $\quad v(x,0) = 0 \quad$ for $0 \leq x \leq \pi,$

(4') $\quad v_t(x,0) = 0 \quad$ for $0 \leq x \leq \pi.$

By part (b), the energy function of v,

$$E(t) = \frac{1}{2}\int_0^\pi \left[v_t^2(x,t) + v_x^2(x,t) \right] dx$$

by (3')

$$v_x(x,0) = \lim_{h \to 0} \frac{\overbrace{v(x+h,0)}^{0} - \overbrace{v(x,0)}^{0}}{h} \equiv 0$$

is decreasing on $[0,\infty)$. Hence, for all $t > 0$,

Identically zero by (3') and (4').

$$0 \leq E(t) \leq E(0) = \frac{1}{2}\int_0^\pi \left[v_t^2(x,0) + v_x^2(x,0) \right] dx = 0$$

$$\therefore \quad E(t) = \frac{1}{2}\int_0^\pi \left[v_t^2(x,t) + v_x^2(x,t) \right] dx = 0 \quad \text{for all } t \geq 0.$$

The vanishing theorem implies $v_t(x,t) = 0 = v_x(x,t)$ for all $0 \leq x \leq \pi$ and $t \geq 0$. Thus $v(x,t) = \text{constant}$ for $0 \leq x \leq \pi$ and $0 \leq t < \infty$, and by (3') the constant must be zero. Thus $u(x,t) - u_1(x,t) = v(x,t) = 0$ for all $0 \leq x \leq \pi$, $0 \leq t < \infty$. That is, $u_1 = u$. Q.E.D.