Mathematics 325 Exam I Name: Dr. Grow

Summer 2000

1.(25 pts.) (a) Find the general solution in the xy-plane of the partial differential equation

(*) $$yu_x - xu_y = 0.$$

(b) Sketch some of the characteristic curves of the pde (*).

(c) Find the solution of the pde (*) which satisfies

$$u(x,0) = x^4 \quad \text{for } -\infty < x < \infty.$$

$[y,-x]\cdot\nabla u = 0$

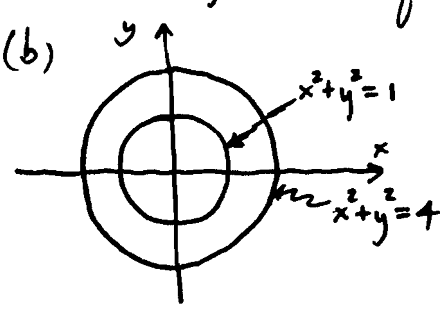
$D_{[y,-x]}u = 0$ $\therefore$ u is constant along (characteristic) curves whose tangent at (x,y) is parallel to $[y,-x]$. Thus the characteristic curves of the pde satisfy $\frac{dy}{dx} = \frac{-x}{y} \Rightarrow y\,dy = -x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$

$\Rightarrow x^2+y^2 = c_1$ $(c_1 = 2c)$. These are circles with center at the origin and radius $\sqrt{c_1}$. Along the characteristic circles, u is constant; so, on such a curve,

$$u(x,y(x)) = u(\sqrt{c_1}, y(\sqrt{c_1})) = u(\sqrt{c_1},0) = f(c_1).$$

+16 (a) Therefore, the general solution to (*) in the xy-plane is $\boxed{u(x,y) = f(x^2+y^2)}$ where f is a C^1-function of a single real variable.

+3 (b)

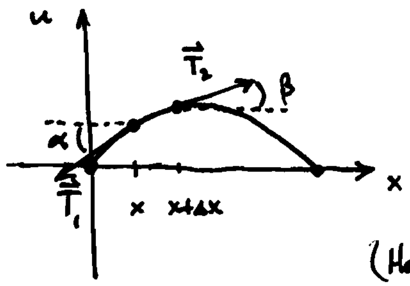


+6 (c) $x^4 = u(x,0) = f(x^2+0^2) = f(x^2) \Rightarrow f(t) = t^2$ for all $t \geq 0$.

Hence the solution of (*) satisfying the condition $u(x,0) = x^4$ for $-\infty < x < \infty$ is $\boxed{u(x,y) = (x^2+y^2)^2}$.

Notation: $u(x,t)$ = vertical displacement of string at position x and time t

2. (25 pts.) An elastic string is held fixed at the endpoints and plucked. Carefully derive the partial differential equation that governs the motion of the string, assuming that air exerts a resistance at each point of the string which is proportional to the string's velocity at that point.



Fix $t>0$ and consider the segment of string between x and $x+\Delta x$. Applying Newton's second law of motion ($\vec{F}=m\vec{a}$) to this segment yields

$$\text{(Horizontal)} \qquad 0 = -|\vec{T}_1|\cos(\alpha) + |\vec{T}_2|\cos(\beta) \qquad (1)$$

10 pts. to here

$$\text{(Vertical)} \qquad \int_x^{x+\Delta x} \rho u_{tt}(\xi,t)\,d\xi = -|\vec{T}_1|\sin(\alpha) + |\vec{T}_2|\sin(\beta) - \int_x^{x+\Delta x} k u_t(\xi,t)\,d\xi$$

where ρ is the constant mass density of the string and k is a positive proportionality constant. Observe that $\tan(\alpha)$ = slope of the string at position x = $u_x(x,t)$. Thus $\sin(\alpha) = \dfrac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$ and $\cos(\alpha) = \dfrac{1}{\sqrt{1+u_x^2(x,t)}}$. (Similar expressions hold for β.) Since the string's deflections are small, $u_x^2 \ll 1$. Using this approximation in (1) yields

$$0 = -|\vec{T}_1|\cdot 1 + |\vec{T}_2|\cdot 1 \quad \Rightarrow \quad |\vec{T}| = T = \text{constant (throughout string)}.$$

Also, this approximation yields in (2):

$$\int_x^{x+\Delta x} \rho u_{tt}(\xi,t)\,d\xi = T\left[u_x(x+\Delta x,t) - u_x(x,t)\right] - \int_x^{x+\Delta x} k u_t(\xi,t)\,d\xi .$$

20 pts. to here

Dividing by Δx and letting $\Delta x \to 0$ produces

$$\rho u_{tt}(x,t) = T u_{xx}(x,t) - k u_t(x,t);$$

I.e. $\boxed{\rho u_{tt} - T u_{xx} + k u_t = 0}$.

(If we don't use the approximation $u_x^2 \ll 1$ and have a variable density $\rho = \rho(x)$, then the equation of motion is: $\rho(x) u_{tt}(x,t) - \dfrac{|\vec{T}(x,t,u)|}{\sqrt{1+u_x^2(x,t)}} u_{xx}(x,t) + k u_t(x,t) = 0$.)

3.(25 pts.) Classify the following partial differential equations as hyperbolic, parabolic, or elliptic, and if possible, find the general solution in the xy-plane.

(a) $u_{xx} + 4u_{yy} - 4u_{xy} + 25u = 0$

(b) $u_{xx} + u_{xy} + 3u_{yy} + u_{yx} = 0$

5 pts. (a) $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$ $\boxed{\text{parabolic}}$

The pde is expressible as $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 u + 25u = 0$.

Let $\xi = 2x + y$, Then the chain rule implies

$\eta = -x + 2y$. $\frac{\partial}{\partial x} = 2\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$.

Consequently, $\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} = \cancel{2\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta} - 2\left(\cancel{\frac{\partial}{\partial \xi}} + 2\frac{\partial}{\partial \eta}\right) = -5\frac{\partial}{\partial \eta}$. Therefore,

the pde (a) is equivalent to $\left(-5\frac{\partial}{\partial \eta}\right)^2 u + 25u = 0$

$$\Rightarrow \cancel{25}\frac{\partial^2 u}{\partial \eta^2} + \cancel{25}u = 0.$$

By ODE methods, we see that $\cos(\eta)$ and $\sin(\eta)$ are fundamental solutions of this equation. Consequently, the solution in the $\xi\eta$-plane is

$$u = c_1(\xi)\cos(\eta) + c_2(\xi)\sin(\eta)$$

Thus, the general solution of (a) in the xy-plane is

5 pts. $$\boxed{u(x,y) = f(2x+y)\cos(2y-x) + g(2x+y)\sin(2y-x)}$$

where f and g are arbitrary C^2-functions of a single real variable.

5 pts. (b) $B^2 - 4AC = 2^2 - 4(1)(3) = -8 < 0$ $\boxed{\text{elliptic}}$

4.(25 pts.) (a) Derive the general solution to

(*) $$u_{tt} = c^2 u_{xx}$$

in the xt-plane.

(b) Derive a formula for the solution to (*) which satisfies the initial conditions

(**) $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$

for $-\infty < x < \infty$. (Here ψ and ϕ are prescribed "sufficiently smooth" functions of a single real variable.)

(c) Write, and simplify as much as possible, the solution to $u_{tt} = u_{xx}$ in the xt-plane which satisfies $u(x,0) = e^{-x^2}$ and $u_t(x,0) = 2xe^{-x^2}$ for $-\infty < x < \infty$.

(d) Sketch profiles of the solution to part (c) for times t = 1, t = 2, and t = 3.

BONUS(10 pts.). Derive a general (nontrivial) relation between ϕ and ψ which will produce a solution to (*)-(**) consisting solely of a wave traveling to the right along the x-axis.

(a) (*) can be expressed as $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$.

Let $\xi = ct + x$, $\eta = ct - x$. Then the chain rule implies $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$. Hence

$$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = \cancel{c\frac{\partial}{\partial \xi}} + c\frac{\partial}{\partial \eta} - c\left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right) = 2c\frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = c\frac{\partial}{\partial \xi} + \cancel{c\frac{\partial}{\partial \eta}} + c\left(\frac{\partial}{\partial \xi} - \cancel{\frac{\partial}{\partial \eta}}\right) = 2c\frac{\partial}{\partial \xi}.$$

Consequently, (*) is equivalent to $\left(2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u = 0$

$$\Rightarrow \quad \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0.$$

Thus, $\frac{\partial u}{\partial \xi} = c_1(\xi)$ and $u = \int c_1(\xi)d\xi + c_2(\eta) = f(\xi) + g(\eta)$.

7 pts. In the xt-plane, the general solution of (*) is $\boxed{u(x,t) = f(x+ct) + g(x-ct)}$ where f and g are arbitrary C^2-functions of a single real variable.

(Note, for future reference, that $u_t(x,t) = cf'(x+ct) - cg'(x-ct)$.)

(b) Apply the I.C.'s (**) to the general solution of (*) in the xt-plane to get:

$$(1) \qquad \varphi(x) = u(x,0) = f(x+0) + g(x-0) = f(x) + g(x) \quad \text{for } -\infty < x < \infty.$$

$$(2) \qquad \psi(x) = u_t(x,0) = cf'(x+0) - cg'(x-0) = cf'(x) - cg'(x) \quad \text{"}.$$

Differentiating (1) gives

$$(1') \qquad \varphi'(x) = f'(x) + g'(x) \quad \text{for } -\infty < x < \infty.$$

Multiply (1') by c and subtract from (2) to get $-c\varphi'(x) + \psi(x) = -2cg'(x)$.
Integrating over the interval $0 \le \xi \le x$ and simplifying gives

$$g(x) - g(0) = \int_0^x g'(\xi)d\xi = \int_0^x \left[\tfrac{1}{2}\varphi'(\xi) - \tfrac{1}{2c}\psi(\xi)\right]d\xi = \tfrac{1}{2}\varphi(x) - \tfrac{1}{2}\varphi(0) - \tfrac{1}{2c}\int_0^x \psi(\xi)d\xi$$

Substituting in (1) yields

$$f(x) = \varphi(x) - g(x) = \tfrac{1}{2}\varphi(x) + \tfrac{1}{2}\varphi(0) + \tfrac{1}{2c}\int_0^x \psi(\xi)d\xi - g(0).$$

Thus

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \tfrac{1}{2}\varphi(x+ct) + \tfrac{1}{2}\varphi(0) + \tfrac{1}{2c}\int_0^{x+ct} \psi(\xi)d\xi - g(0) \\ &\quad + \tfrac{1}{2}\varphi(x-ct) - \tfrac{1}{2}\varphi(0) - \tfrac{1}{2c}\int_0^{x-ct}\psi(\xi)d\xi + g(0) \end{aligned}$$

7 pts. i.e. $$\boxed{u(x,t) = \tfrac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi}$$

(c) $c=1$, $\varphi(x) = e^{-x^2}$, and $\psi(x) = 2xe^{-x^2}$. Thus, by (b),

$$u(x,t) = \tfrac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \tfrac{1}{2}\int_{x-t}^{x+t} 2\xi e^{-\xi^2}d\xi$$

Substitution: $v = -\xi^2$, $dv = -2\xi d\xi$

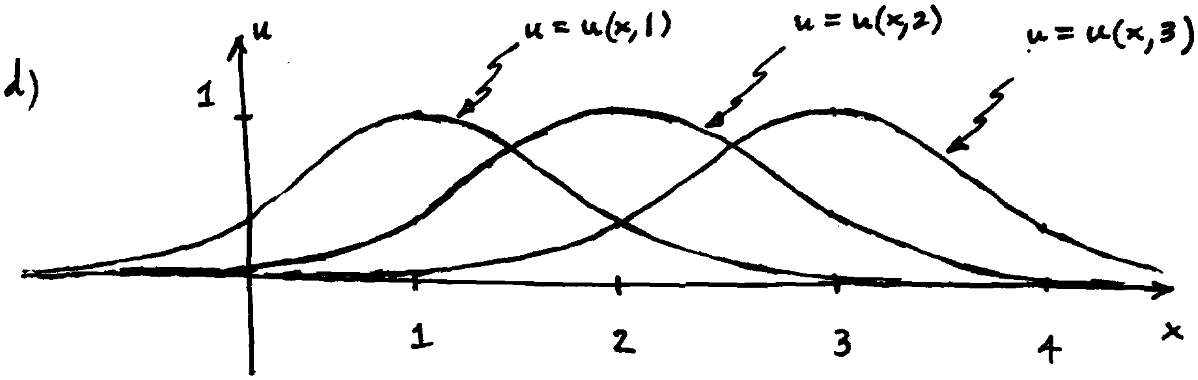
$$= \tfrac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] - \tfrac{1}{2}\left(e^{-\xi^2}\right)\Big|_{\xi=x-t}^{x+t}$$

$$= \tfrac{1}{2}\left[\cancel{e^{-(x+t)^2}} + e^{-(x-t)^2}\right] - \tfrac{1}{2}\left[\cancel{e^{-(x+t)^2}} - e^{-(x-t)^2}\right]$$

(OVER)

8 pts 4(c) (cont.) $\boxed{u(x,t) = e^{-(x-t)^2}}$

3 pts. (d)



BONUS: In order to obtain a solution to (*)–(**) which consists solely of a wave traveling to the right, we must have

$$\text{constant} = f(x+ct) = \tfrac{1}{2}\varphi(x+ct) + \tfrac{1}{2}\varphi(0) + \frac{1}{2c}\int_0^{x+ct}\psi(\xi)\,d\xi - g(0)$$

for all $-\infty<x<\infty$ and all $t\geq 0$. Thus

$$0 = f'(x) = \tfrac{1}{2}\varphi'(x) + \frac{1}{2c}\psi(x) \quad \text{for all } -\infty<x<\infty,$$

or $\boxed{\psi(x) = -c\varphi'(x) \quad \text{for all } -\infty<x<\infty}$.

In this case,

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)\,d\xi \\ &= \tfrac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}-c\varphi'(\xi)\,d\xi \\ &= \tfrac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] - \tfrac{1}{2}\left[\varphi(x+ct)-\varphi(x-ct)\right] \\ &= \varphi(x-ct) \end{aligned}$$

$\leftarrow$ A wave traveling to the right along the x-axis with speed c.

EI

$\mu = 66.2$

$\sigma = 23.4$

87–100	𝍸 |	6
73–86	|	1
60–72	||||	4
50–59	𝍸	5
0–49	𝍸	5