Mathematics 315	Final Exam, Part I Spring 2007	Name: Dr. Grow (1 pt.)

This portion of the 200-point final examination is closed book/notes. You are to turn in your solutions to this portion before receiving the second part.

1.(30 pts.) (a) State Lebesgue's Monotone Convergence Theorem.
(b) Give an example to show that the conclusion of the Monotone Convergence Theorem need not hold for a pointwise decreasing sequence of nonnegative measurable functions.
(c) State Lebesgue's Dominated Convergence Theorem.
(d) Give an example to show that the conclusion of the Dominated Convergence Theorem need not hold for a pointwise convergent sequence of integrable functions whose limit is an integrable function.
(e) State Fatou's Lemma.
(f) Use the Monotone Convergence Theorem to prove Fatou's Lemma.

2.(30 pts.) (a) State Littlewood's Three Principles.
(b) State a rigorous version for each one of Littlewood's principles.

3.(30 pts.) In each of the following, compute the Lebesgue integral of f over the set E or show that f is not integrable over E. Please justify the steps in your computations.

(a) $f(x) = \begin{cases} 5 & \text{if } x \in \mathbb{Q}, \\ -2 & \text{if } x \in [-1,0] \setminus \mathbb{Q}, \\ 3 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$ $\qquad E = [-1,1].$

(b) $f(x) = \dfrac{1}{\sqrt[3]{x}},$ $\qquad E = [0,1].$

(c) $f(x) = \begin{cases} e^{(i+1)|x|} & \text{if } x \in \mathbb{A}, \\ e^{(i-1)|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{A}. \end{cases}$ $\qquad E = (-\infty, \infty).$

#1 (a) (MCT) Let $f_n : E \to \mathbb{R}$ $(n=1,2,3,\dots)$ be a sequence of measurable

pts. functions such that $0 \le f_1(x) \le f_2(x) \le \dots$ for every $x \in E$. Then

$$\lim_{n\to\infty} \int_E f_n \, dx = \int_E \left(\lim_{n\to\infty} f_n\right) dx .$$

5 pts. (b) Let $f_n = \chi_{(n,\infty)}$ $(n=1,2,3,\dots)$. Then $0 \le f_{n+1}(x) \le f_n(x)$ for all $x \in (0,\infty)$ and all $n \ge 1$ yet

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n \, dx = \infty \ne 0 = \int_{(0,\infty)} 0 \, dx = \int_{(0,\infty)} \left(\lim_{n\to\infty} f_n\right) dx .$$

5 pts. (c) (DCT) Let $f_n : E \to [-\infty, \infty]$ $(n=1,2,3,\dots)$ be a sequence of measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists a.e. in E. If there exists a function $g \in L^1(E)$ such that $|f_n(x)| \le g(x)$ a.e. in E for all $n \ge 1$, then $f \in L^1(E)$ and $\int_E f \, dx = \lim_{n\to\infty} \int_E f_n \, dx$.

5 pts (d) Let $f_n = \chi_{(n,n+1)}$ $(n=1,2,3,\dots)$. Then $\lim_{n\to\infty} f_n = 0$ on $(0,\infty)$ and each $f_n \in L^1(0,\infty)$ with $\int_{(0,\infty)} f_n \, dx = 1$ $(n=1,2,3,\dots)$, so

$$\int_{(0,\infty)} \left(\lim_{n\to\infty} f_n\right) dx = \int_{(0,\infty)} 0 \, dx = 0 \ne 1 = \lim_{n\to\infty} \int_{(0,\infty)} f_n \, dx .$$

#2(a) Littlewood's Three Principles are:

+4 1. every measurable set is nearly a finite union of intervals;

+4 2. every measurable function is nearly continuous;

+4 3. every convergent sequence of measurable functions is nearly uniformly convergent.

Here are rigorous versions for Littlewood's Three Principles.

+6 1. Let $E \subseteq \mathbb{R}$ with $m(E) < \infty$. Then to each $\varepsilon > 0$ there corresponds a finite union U_ε of open intervals such that $m(U_\varepsilon \triangle E) < \varepsilon$.

+6 2. (Lusin) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then to each $\delta > 0$ there corresponds a continuous function $\varphi_\delta : \mathbb{R} \to \mathbb{R}$ such that $m\left(\left\{x \in \mathbb{R} : \varphi_\delta(x) \neq f(x)\right\}\right) < \delta$.

+6 3. (Egoroff) Let $E \subseteq \mathbb{R}$ with $m(E) < \infty$ and let $f_n : E \to \mathbb{R}$ $(n = 1, 2, 3, \ldots)$ be a sequence of measurable functions which is pointwise convergent a.e. on E. Then to each $\varepsilon > 0$ there corresponds $A_\varepsilon \subseteq E$ such that $m(A_\varepsilon) < \varepsilon$ and $\langle f_n \rangle_{n=1}^{\infty}$ is uniformly convergent on $E \setminus A_\varepsilon$.

#3. (a) $f = 5\chi_{\mathbb{Q}} - 2\chi_{[-1,0]\setminus\mathbb{Q}} + 3\chi_{[0,1]\setminus\mathbb{Q}}$ so

10pts.

$$\int_{[-1,1]} f\,dx = 5m\big(\mathbb{Q}\cap[-1,1]\big) - 2m\Big(\big([-1,0]\setminus\mathbb{Q}\big)\cap[-1,1]\Big) + 3m\Big(\big([0,1]\setminus\mathbb{Q}\big)\cap[-1,1]\Big)$$

$$= 5\cdot 0 - 2\cdot 1 + 3\cdot 1$$

$$= \boxed{1}$$

10pts. (b) Let $f_n(x) = \min\left\{\frac{1}{\sqrt[3]{x}}, n\right\}$ for $x\in[0,1]$ and $n = 1,2,3,\ldots$

Then $\langle f_n\rangle_{n=1}^{\infty}$ is a sequence of continuous (and hence measurable) functions on $[0,1]$ such that $0 \le f_1(x) \le f_2(x) \le \ldots$ for $x\in[0,1]$. The Monotone Convergence Theorem implies

$$\int_{[0,1]} \frac{1}{\sqrt[3]{x}}\,dx = \int_{[0,1]} \Big(\lim_{n\to\infty} f_n\Big)dx = \lim_{n\to\infty}\int_{[0,1]} f_n\,dx .$$

But $f_n(x) = \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n^3}, \\ \frac{1}{\sqrt[3]{x}} & \text{if } \frac{1}{n^3} < x \le 1, \end{cases}$ $(n = 1,2,3,\ldots)$

is (bounded and) Riemann integrable on $[0,1]$ so

$$\int_{[0,1]} f_n\,dx = \int_0^1 f_n\,dx = \int_0^{1/n^3} n\,dx + \int_{1/n^3}^{1} \frac{1}{\sqrt[3]{x}}\,dx = \frac{1}{n^2} + \frac{3}{2}x^{2/3}\Big|_{1/n^3}^{1}$$

$$= \frac{1}{n^2} + \frac{3}{2} - \frac{3}{2n^2} \longrightarrow \frac{3}{2} \quad \text{as } n \to \infty.$$

Therefore $\displaystyle\int_{[0,1]} \frac{1}{\sqrt[3]{x}}\,dx = \boxed{\frac{3}{2}}$.

10 pts. → (c) Since A is a countable subset of $\mathbb{R}$, $m(A) = 0$.

Consequently, $f(x) = e^{(i-1)|x|}$ a.e. on $\mathbb{R}$ so $\displaystyle\int_{(-\infty,\infty)} f\,dx = \int_{(-\infty,\infty)} e^{(i-1)|x|}\,dx$.

Let $f_n(x) = e^{(i-1)|x|}\chi_{(-n,n)}(x)$ for x in $(-\infty,\infty)$ and $n = 1,2,3,\ldots$

Since $|f_n(x)| \le e^{-|x|}$ for all x in $(-\infty,\infty)$ and all $n \ge 1$, $\lim_{n\to\infty} f_n(x) = e^{(i-1)|x|}$ on $(-\infty,\infty)$, and the function $x \mapsto e^{-|x|}$ belongs to $L^1(-\infty,\infty)$, the Dominated Convergence theorem implies

$$\int_{(-\infty,\infty)} e^{(i-1)|x|}\,dx = \lim_{n\to\infty} \int_{(-\infty,\infty)} f_n\,dx$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{(i-1)|x|}\,dx$$

$$= \lim_{n\to\infty} 2\int_0^n e^{(i-1)x}\,dx$$

$$= \lim_{n\to\infty} \left.\frac{2e^{(i-1)x}}{i-1}\right|_{x=0}^{n}$$

$$= \lim_{n\to\infty} \left(\frac{2e^{(i-1)n}}{i-1} - \frac{2}{i-1}\right) = \frac{2}{1-i} = \boxed{1+i}.$$