Mathematics 325 Final Exam Name: Dr. Grow
Summer 2000

This final exam consists of seven problems of equal value. Please choose how much you want this exam to count by circling one of the following statements.

I want this exam to count 200 points.
I want this exam to count 300 points.

1. Classify the following partial differential equations as hyperbolic, parabolic, or elliptic, and if possible, find the general solution in the xy-plane.

15 pts — (a) $u_{xx} - 3u_{xy} - 4u_{yy} = 0$ $B^2 - 4AC = 9 - 4(1)(-4) > 0$ hyperbolic +1

15 pts — (b) $u_{xx} + u_{yy} + 2u_{xy} + 36u = 0$ $B^2 - 4AC = 4 - 4(1)(1) = 0$ parabolic +1

2 (a) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right)u = 0$ Let $\xi = 4x + y$ 4

$\eta = x - y$

6 $\left(\cancel{5}\frac{\partial}{\partial \xi}\right)\left(\cancel{5}\frac{\partial}{\partial \eta}\right)u = 0$ Then $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

8 $\frac{\partial u}{\partial \eta} = c_1(\eta)$ and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$.

10 $u = \int c_1(\eta)\,d\eta + c_2(\xi)$ $\therefore$ $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 5\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y} = 5\frac{\partial}{\partial \eta}$.

12 $u = f(\eta) + g(\xi)$

14 $\boxed{u = f(x-y) + g(4x+y)}$ where f and g are arbitrary C^2 functions of a single real variabl

2 (b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u + 36u = 0$ Let $\xi = x - y$ 4

$\eta = x + y$

6 $\left(2\frac{\partial}{\partial \eta}\right)^2 u + 36u = 0$ Then $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

$\frac{\partial^2 u}{\partial \eta^2} + 9u = 0$ and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$.

12 $u = c_1(\xi)\cos(3\eta) + c_2(\xi)\sin(3\eta)$ $\therefore$ $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta}$.

14 $\boxed{u = f(x-y)\cos(3(x+y)) + g(x-y)\sin(3(x+y))}$ where f and g are arbitrary C^2 functions of a single real variable.

2. (a) Write, and simplify as much as possible, the solution to $u_{tt} = u_{xx}$ in the xt-plane which satisfies

$$u(x,0) = e^{-x^2} \quad \text{and} \quad u_t(x,0) = -2xe^{-x^2} \quad \text{for } -\infty < x < \infty.$$

(b) Sketch profiles of the solution to part (a) for times $t = 1$, $t = 2$, and $t = 3$.

(c) Derive a general (nontrivial) relation between ϕ and ψ which will produce a solution to $u_{tt} = u_{xx}$ in the xt-plane satisfying

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty$$

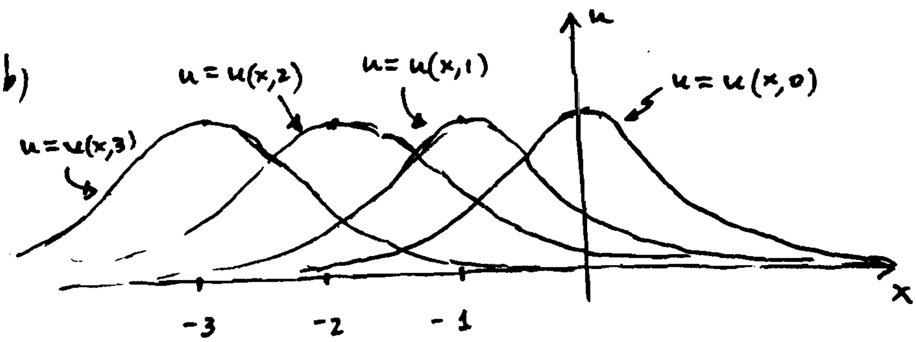
and such that u consists solely of a wave traveling to the left along the x-axis.

13 (a) $$u(x,t) = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} -2\xi e^{-\xi^2}\,d\xi \qquad \text{(d'Alembert)}$$

$$= \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2}\left(e^{-\xi^2}\right)\Big|_{\xi=x-t}^{x+t}$$

$$= \boxed{e^{-(x+t)^2}}$$

4 (b)



13 (c) $$u(x,t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\,d\xi$$

$$= \frac{1}{2}\left[\varphi(x+t) + \int_0^{x+t}\psi(\xi)\,d\xi\right] + \underbrace{\frac{1}{2}\left[\varphi(x-t) + \int_{x-t}^{0}\psi(\xi)\,d\xi\right]}_{\text{We need this to vanish identically.}}$$

$$\therefore \quad \varphi(\xi) + \int_{\xi}^{0}\psi(\eta)\,d\eta = 0 \quad \text{for all } -\infty < \xi < \infty.$$

$$\Rightarrow \quad \boxed{\varphi'(\xi) = \psi(\xi)} \quad \text{for all } -\infty < \xi < \infty.$$

3. Find the solution to

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

which satisfies

$$u(x,0) = x^3 \quad \text{for } -\infty < x < \infty.$$

You may find the following identities useful:

$$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} p e^{-p^2}\,dp = 0, \qquad \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}, \qquad \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp = 0.$$

6 $$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}\, y^3\,dy = \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}} (x + p\sqrt{4t})^3\,dp$$ 12

Let $p = (y-x)/\sqrt{4t}$.
Then $dp = dy/\sqrt{4t}$.

18 $$= \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}} \left(x^3 + 3x^2 p\sqrt{4t} + 3x(p\sqrt{4t})^2 + (p\sqrt{4t})^3 \right) dp$$

24 $$= \frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{3x^2\sqrt{4t}}{\sqrt{\pi}} \cancelto{0}{\int_{-\infty}^{\infty} e^{-p^2} p\,dp} + \frac{12xt}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp + \frac{4t\sqrt{4t}}{\sqrt{\pi}} \cancelto{0}{\int_{-\infty}^{\infty} e^{-p^2} p^3\,dp}$$

30 $$= \boxed{x^3 + 6xt}$$

Check: $u_t - u_{xx} = 6x - 6x \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^3$

4. (a) Let f and ψ be piecewise continuous, absolutely integrable functions on $(-\infty,\infty)$. Use Fourier transform methods to solve

24 $$u_{xx} + u_{yy} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the boundary condition

$$u(x,0) = f(x) \quad \text{for } -\infty < x < \infty$$

and the decay conditions

$$\lim_{y\to\infty} u(x,y) = 0 \quad \text{for each } x \in (-\infty,\infty)$$

and, for each $y > 0$,

$$|u(x,y)| \le |\psi(x)| \quad \text{for all } -\infty < x < \infty.$$

6 (b) Compute an explicit formula for the solution in part (a) if the function f is given by $f(x) = 1$ for $|x| < 1$, and $f(x) = 0$ otherwise.

(a) $\mathcal{F}(u_{xx} + u_{yy})(\xi) = \mathcal{F}(0)(\xi)$

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) + (i\xi)^2\mathcal{F}(u)(\xi) = 0$$

$$\mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}$$

If $\xi > 0$ then $0 = \lim_{y\to\infty}\mathcal{F}(u)(\xi)$ so $c_1(\xi) = 0$

If $\xi < 0$ then $0 = \lim_{y\to\infty}\mathcal{F}(u)(\xi)$ so $c_2(\xi) = 0$

$$\therefore \mathcal{F}(u)(\xi) = \begin{cases} c_2(\xi)e^{-\xi y} & \text{if } \xi > 0, \\ c_1(\xi)e^{\xi y} & \text{if } \xi < 0, \end{cases}$$

$$= c(\xi)e^{-|\xi|y}$$

$$\mathcal{F}(f)(\xi) = \mathcal{F}(u)(\xi)\Big|_{y=0} = c(\xi)e^{-|\xi|y}\Big|_{y=0} = c(\xi)$$

$$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)e^{-|\xi|y}$$

From Table of Fourier Transforms, entry C (with $a = y$) we have

$$e^{-|\xi|y} = \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

$$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$\therefore \mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f * \frac{\sqrt{2}\,y}{\sqrt{\pi}\,((\cdot)^2+y^2)}\right)(\xi)$$

By the Fourier inversion theorem,

$$u(x,y) = \frac{1}{\pi}\left(f * \frac{y}{(\cdot)^2+y^2}\right)(x)$$

$$= \boxed{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)\,y\,ds}{(x-s)^2+y^2}}$$ 2

(b) $$u(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{y\,ds}{(x-s)^2+y^2}$$

Let $z = \frac{s-x}{y}$. Then $dz = \frac{ds}{y}$

$$u(x,y) = \frac{1}{\pi}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}}\frac{\cancel{y}\,dz}{\cancel{y}(z^2+1)}$$

$$= \frac{1}{\pi}\left[\text{Arctan}\left(\frac{1-x}{y}\right) - \text{Arctan}\left(\frac{-1-x}{y}\right)\right]$$

$$= \boxed{\frac{1}{\pi}\left[\text{Arctan}\left(\frac{1+x}{y}\right) + \text{Arctan}\left(\frac{1-x}{y}\right)\right]}$$

5. Solve $\nabla^2 u = 0$ in the unit cube C: $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, subject to the boundary conditions

$$u(x,y,0) = \sin(\pi x)\sin^3(\pi y) \quad \text{for } 0 \le x \le 1,\ 0 \le y \le 1,$$

and $u = 0$ on the other five faces of the cube C.

$$u(x,y,z) = X(x)Y(y)Z(z), \quad u_{xx}+u_{yy}+u_{zz}=0 \Rightarrow X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \text{constant} = \lambda \Rightarrow -\frac{Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda = \text{constant} = \mu$$

$$\therefore\ X''(x) + \lambda X(x) = 0,\quad X(0) = X(1) = 0$$

$$Y''(y) + \mu Y(y) = 0,\quad Y(0) = Y(1) = 0$$

10 $$Z''(z) - (\lambda+\mu)Z(z) = 0,\quad Z(1) = 0$$

B.C.'s on Faces $x=0, x=1 \Rightarrow X(0)Y(y)Z(z) = 0$ and $X(1)Y(y)Z(z) = 0$ for $0 \le y \le 1$ and $0 \le z \le 1$

Nontriviality of solution $u = XYZ \Rightarrow X(0) = X(1) = 0$.

B.C.'s on faces $y=0, y=1$, and $z=1$ similarly yield $Y(0) = Y(1) = 0 = Z(1)$.

Eigenvalues	Eigenfunctions	
14 $\lambda_l = (l\pi)^2$	$X_l(x) = \sin(l\pi x)$	$(l = 1,2,3,\ldots)$
18 $\mu_m = (m\pi)^2$	$Y_m(y) = \sin(m\pi y)$	$(m = 1,2,3,\ldots)$

Solution to the z-equation when $\lambda = \lambda_l$ and $\mu = \mu_m$: $Z_{l,m}(z) = A\cosh(\pi z\sqrt{l^2+m^2}) + B\sinh(\pi z\sqrt{l^2+m^2})$

$Z_{l,m}(1) = 0 \Rightarrow Z_{l,m}(z) = \sinh(\pi(1-z)\sqrt{l^2+m^2})$ (up to a constant multiple)

Formal solution to the homogeneous part of the problem:

22 $$u(x,y,z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{l,m}\sin(l\pi x)\sin(m\pi y)\sinh(\pi(1-z)\sqrt{l^2+m^2})$$

We want to choose the coefficients $B_{l,m}$ $(l = 1,2,3,\ldots, m = 1,2,3,\ldots)$ so the nonhomogeneous B.C. is met:

24 $$\sin(\pi x)\sin^3(\pi y) = u(x,y,0) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{l,m}\sin(l\pi x)\sin(m\pi y)\sinh(\pi\sqrt{l^2+m^2}) \quad \begin{pmatrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{pmatrix}$$

$$\text{But } \sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{e^{3i\theta} - 3e^{2i\theta}\cdot e^{-i\theta} + 3e^{i\theta}\cdot e^{-2i\theta} - e^{-3i\theta}}{-8i} = -\frac{1}{4}\underbrace{\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)}_{\sin(3\theta)} + \frac{3}{4}\underbrace{\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)}_{\sin(\theta)}$$

$$\therefore\ \sin(\pi x)\sin^3(\pi y) = \frac{3}{4}\sin(\pi x)\sin(\pi y) - \frac{1}{4}\sin(\pi x)\sin(3\pi y).$$

28 Choose $B_{1,1}\sinh(\pi\sqrt{2}) = \frac{3}{4}$, $B_{1,3}\sinh(\pi\sqrt{10}) = -\frac{1}{4}$, and all other $B_{l,m} = 0$.

30 $$\therefore\ \boxed{u(x,y,z) = \frac{3}{4\sinh(\pi\sqrt{2})}\sin(\pi x)\sin(\pi y)\sinh(\pi(1-z)\sqrt{2}) - \frac{1}{4\sinh(\pi\sqrt{10})}\sin(\pi x)\sin(3\pi y)\sinh(\pi(1-z)\sqrt{10})}$$

2d 6. (a) Solve $\nabla^2 u = 0$ in the unit disk $0 \le r < 1$ subject to the boundary condition $u(1;\theta) = 1$ if $0 < \theta < \pi$ and $u(1;\theta) = 0$ if $-\pi < \theta < 0$.

(b) What is the value of the solution to (a) at the center of the disk? Justify your answer.

12 (a) Poisson's Formula: $$u(r;\theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{h(\varphi)\,d\varphi}{a^2 - 2ar\cos(\theta-\varphi) + r^2} \qquad \begin{pmatrix} 0 \le r < a \\ -\pi \le \theta < \pi \end{pmatrix}$$

In our case, $a = 1$ and $h(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } -\pi < \theta < 0. \end{cases}$

18 $$\therefore \boxed{u(r;\theta) = \frac{1-r^2}{2\pi} \int_0^{\pi} \frac{d\varphi}{1 - 2r\cos(\theta-\varphi) + r^2}} \qquad \begin{pmatrix} 0 \le r < 1 \\ -\pi \le \theta < \pi \end{pmatrix}$$

Using the integral formula $$\int \frac{d\psi}{A + B\cos(\psi)} = \frac{2}{\sqrt{A^2 - B^2}} \operatorname{Arctan}\left(\frac{\tan(\psi/2)\sqrt{A^2-B^2}}{A+B} \right)$$

(which can be derived using the Weierstrass substitution $z = \tan(\frac{\psi}{2})$), we have

$$u(r;\theta) = \frac{1-r^2}{2\pi} \int_{-\theta}^{\pi-\theta} \frac{d\psi}{1 + r^2 - 2r\cos(\psi)}$$

$$= \frac{1-r^2}{2\pi} \cdot \frac{2}{\sqrt{(1-r^2)^2}} \operatorname{Arctan}\left[\frac{\tan(\psi/2)\sqrt{(1-r^2)^2}}{(1-r)^2} \right] \Bigg|_{\psi=-\theta}^{\pi-\theta}$$

$$\begin{aligned} A+B &= 1 + r^2 - 2r = (1-r)^2 \\ A^2 - B^2 &= (1+r^2)^2 - 4r^2 \\ &= 1 - 2r^2 + r^4 \\ &= (1-r^2)^2 \end{aligned}$$

$$= \frac{1}{\pi} \left\{ \operatorname{Arctan}\left[\frac{\tan(\frac{\pi-\theta}{2})\cancel{(1-r)}(1+r)}{(1-r)^{\cancel{2}}} \right] - \operatorname{Arctan}\left[\frac{\tan(-\theta/2)\cancel{(1-r)}(1+r)}{(1-r)^{\cancel{2}}} \right] \right\}$$

24 $$= \boxed{\frac{1}{\pi} \left\{ \operatorname{Arctan}\left[\frac{\cot(\theta/2)(1+r)}{1-r} \right] + \operatorname{Arctan}\left[\frac{\tan(\theta/2)(1+r)}{1-r} \right] \right\}}$$

(b) By the mean-value property for harmonic functions,

6 $$u(0;\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(1;\varphi)\,d\varphi = \frac{1}{2\pi} \int_0^{\pi} 1 \cdot d\varphi = \boxed{\frac{1}{2}}.$$

Alternate solution to #6.

12 (a) $$u(r;\theta) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{in\theta} \left(\frac{r}{a}\right)^{|n|}$$

In our case, $a=1$ and $h(\theta) = \begin{cases} 1 & \text{if } 0<\theta<\pi, \\ 0 & \text{if } -\pi<\theta<0. \end{cases}$

$$\hat{h}(n) = \frac{\langle h, e^{in(\cdot)}\rangle}{\langle e^{in(\cdot)}, e^{in(\cdot)}\rangle} = \frac{\int_{-\pi}^{\pi} h(\theta) e^{-in\theta} d\theta}{\int_{-\pi}^{\pi} |e^{in\theta}|^2 d\theta} = \frac{1}{2\pi}\int_0^{\pi} 1\cdot e^{-in\theta} d\theta$$

18 $$\overset{(n\neq 0)}{=} \frac{1}{2\pi}\left(\frac{e^{-in\theta}}{-in}\right)\Bigg|_0^{\pi} = \frac{i}{2\pi n}\Big[\underbrace{e^{-in\pi}}_{(-1)^n} - 1\Big] = \begin{cases} 0 & \text{if } n=2k \text{ is even} \\ \dfrac{-i}{\pi(2k-1)} & \text{if } n=2k-1 \text{ is odd} \end{cases}$$

20 $$\hat{h}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta) d\theta = \frac{1}{2\pi}\int_0^{\pi} 1\, d\theta = \frac{1}{2}$$

$$u(r;\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \left[\hat{h}(n) e^{in\theta} + \hat{h}(-n) e^{-in\theta}\right]$$

24 $$= \boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} r^{2k-1} \frac{\sin((2k-1)\theta)}{2k-1}}$$

6 (b) $$\boxed{u(0;\theta) = \frac{1}{2}}$$

$\varphi'(0) = \varphi'(1) = 0$

7. (a) Show that the Fourier cosine series of the function ϕ given on

9 $[0,1]$ by $\phi(x) = x^2(1-x)^2$ is

$$\frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)}{(m\pi)^4}.$$

$\varphi(x) = x^2 - 2x^3 + x^4$

$\varphi'(x) = 2x - 6x^2 + 4x^3$

$\varphi''(x) = 2 - 12x + 12x^2$

$\varphi'''(x) = -12 + 24x$

$\varphi^{(4)}(x) = 24$

6 (b) Does the Fourier cosine series of ϕ converge to ϕ uniformly on $[0,1]$? Justify your answer.

9 (c) Solve

$$u_{tt} = u_{xx} \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

subject to the homogeneous boundary/initial conditions

$$u_x(0,t) = 0 = u_x(1,t) \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \le t,\ 0 \le x \le 1,$$

and the nonhomogeneous initial condition

$$u(x,0) = x^2(1-x)^2 \quad \text{for } 0 \le x \le 1.$$

6 (d) Is your solution to (c) the only possible one? Give a complete justification of your answer.

(a) $\underset{n\ge 1}{a_n} = \dfrac{\langle \varphi, \cos(n\pi\cdot)\rangle}{\langle \cos(n\pi\cdot), \cos(n\pi\cdot)\rangle} = 2\displaystyle\int_0^1 \overbrace{\varphi(x)}^{u}\overbrace{\cos(n\pi x)dx}^{dV} = 2\cancel{\varphi(x)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1}^{\,0} - \frac{2}{n\pi}\int_0^1 \overbrace{\varphi'(x)}^{u}\overbrace{\sin(n\pi x)dx}^{dV}$

$= \underbrace{\cancel{\frac{-2}{n\pi}\varphi'(x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1}^{\,0}}_{\text{Since } \varphi'(0)=\varphi'(1)=0} + \dfrac{-2}{(n\pi)^2}\displaystyle\int_0^1 \overbrace{\varphi''(x)}^{u}\overbrace{\cos(n\pi x)dx}^{dV} = \cancel{\frac{-2}{(n\pi)^2}\varphi''(x)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1}^{\,0} + \frac{2}{(n\pi)^3}\int_0^1 \overbrace{\varphi'''(x)}^{u}\overbrace{\sin(n\pi x)dx}^{dV}$

$= \dfrac{2}{(n\pi)^3}(24x-12)\left(-\dfrac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1 + \dfrac{2}{(n\pi)^4}\cancel{\displaystyle\int_0^1 24\cos(n\pi x)dx}^{\,0} = \dfrac{24}{(n\pi)^4}\left[(-1)^{n+1}-1\right] = \begin{cases} \dfrac{-48}{(n\pi)^4} & \text{if } n=2m \text{ is even} \\ 0 & \text{if } n=2m \text{ is odd} \end{cases}$

$= \begin{cases} \dfrac{-3}{(m\pi)^4} & \text{if } n=2m \text{ is even,} \\ 0 & \text{if } n=2m+1 \text{ is odd.} \end{cases}$

$a_0 = \dfrac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} = \displaystyle\int_0^1 \varphi(x)dx = \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\Big|_0^1 = \frac{1}{30}$

The Fourier cosine series of φ is

$$a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x) = \boxed{\frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)}{(m\pi)^4}}.$$

(b) Since φ, φ', and φ'' are continuous on $[0,1]$, and φ satisfies the Neumann B.C.'s that generate the orthogonal set $\{\cos(n\pi x)\}_{n=0}^{\infty}$ (i.e. $\varphi'(0) = \varphi'(1) = 0$), it follows that the Fourier cosine series of φ converges uniformly to φ on $[0,1]$.

(c) $u(x,t) = X(x)T(t) \Rightarrow X(x)T''(t) = X''(x)T(t) \Rightarrow -\dfrac{X''(x)}{X(x)} = -\dfrac{T''(t)}{T(t)} = \text{const.} = \lambda$

nontriviality of solns.

$$\therefore \quad X''(x)+\lambda X(x)=0, \quad X'(0)=X'(1)=0$$
$$T''(t)+\lambda T(t)=0, \quad T'(0)=0$$

$\left\{\begin{array}{l} u_x(0,t)=0 \Rightarrow X'(0)T(t)=0 \Rightarrow X'(0)=0 \\ \text{similarly } u_x(1,t)=0 \Rightarrow X'(1)=0 \\ \text{and } u_t(x,0)=0 \Rightarrow T'(0)=0 \end{array}\right.$

<u>Eigenvalues</u>: <u>Eigenfunctions</u>

$\lambda_n=(n\pi)^2 \qquad X_n(x)=\cos(n\pi x) \qquad (n=0,1,2,\dots)$

Solution to the t-problem corresponding to $\lambda=\lambda_n$: $T_n(t)=\cos(n\pi t)$

$\therefore\ u(x,t)=\sum_{n=0}^{\infty} a_n\cos(n\pi x)\cos(n\pi t)$ is a formal solution to the homogeneous part of the problem. We want to choose the coefficients a_n $(n=0,1,2,\dots)$ so the nonhomogeneous I.C. is met:

$$x^2(1-x)^2=u(x,0)=a_0+\sum_{n=1}^{\infty} a_n\cos(n\pi x) \quad \text{for } 0\le x\le 1.$$

By part (a), $a_0=\frac{1}{30}$ and $a_{2m}=\frac{-3}{(m\pi)^4}$, $a_{2m-1}=0$ $\quad (m=1,2,3,\dots)$.

$$\therefore \quad \boxed{u(x,t)=\frac{1}{30}+\sum_{m=1}^{\infty}\frac{-3\cos(2m\pi x)\cos(2m\pi t)}{(m\pi)^4}}\,.$$

(d) The solution in (c) is $\boxed{\text{unique}}$, for suppose $u=u_2(x,t)$ is another solution. Then

$$w(x,t)=\frac{1}{30}+\sum_{m=1}^{\infty}\frac{-3\cos(2m\pi x)\cos(2m\pi t)}{(m\pi)^4}-u_2(x,t)$$

solves

$$w_{tt}=w_{xx} \qquad (0<x<1,\ 0<t<\infty),$$
$$w_x(0,t)=0=w_x(1,t) \qquad (0\le t<\infty),$$
$$w(x,0)=0=w_t(x,0) \qquad (0\le x\le 1).$$

Let $E(t)=\int_0^1\left[\frac{1}{2}w_t^2(x,t)+\frac{1}{2}w_x^2(x,t)\right]dx \qquad (0\le t<\infty)$.

Then

$$E'(t)=\int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2}w_t^2(x,t)+\frac{1}{2}w_x^2(x,t)\right]dx$$
$$=\int_0^1 w_t(x,t)w_{tt}(x,t)\,dx+\int_0^1 w_x(x,t)w_{tx}(x,t)\,dx$$

$$= \int_0^1 \overbrace{w_t(x,t)}^{v}\,\overbrace{w_{xx}(x,t)\,dx}^{dV} + \int_0^1 w_x(x,t)\,w_{tx}(x,t)\,dx$$

$$= \Big[w_t(1,t)\overbrace{w_x(1,t)}^{0} - w_t(0,t)\overbrace{w_x(0,t)}^{0}\Big] - \int_0^1 \cancel{w_x(x,t)\,w_{tx}(x,t)}\,dx + \int_0^1 \cancel{w_x(x,t)\,w_{tx}(x,t)}\,dx$$

$$= 0$$

so $E(t) =$ constant on $[0,\infty)$. But $w(x,0) = 0$ for all $0 \le x \le 1$ implies $w_x(x,0) = 0$.

$$\therefore \quad E(0) = \int_0^1 \Big[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0)\Big]dx = 0,$$

so $E(t) = 0$ for all $0 \le t < \infty$. But the integrand of E is continuous and nonnegative so $\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) = 0$ for all $0 \le x \le 1$ and all $0 \le t < \infty$. Consequently $w_t(x,t) = w_x(x,t) = 0$ so $w(x,t) =$ constant on $0 \le x \le 1$, $0 \le t < \infty$. However the I.C. $w(x,0) = 0$ $(0 \le x \le 1)$ implies $w(x,t) = 0$ on $0 \le x \le 1$, $0 \le t < \infty$, i.e.

$$u_2(x,t) = \frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)\cos(2\pi m t)}{(m\pi)^4}$$

for all $0 \le x \le 1$ and $0 \le t < \infty$.

Alternate Work for #7(a).

$$\varphi(x) = x^2 - 2x^3 + x^4$$

$$\int_0^1 x^2\cos(n\pi x)\,dx = \cancelto{0}{\left.\frac{x^2\sin(n\pi x)}{n\pi}\right|_0^1} - \frac{2}{n\pi}\int_0^1 x\,\sin(n\pi x)\,dx$$

$$= -\frac{2}{n\pi}\left[-x\frac{\cos(n\pi x)}{n\pi}\Big|_0^1 + \cancelto{0}{\int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx}\right]$$

$$= \frac{2}{(n\pi)^2}\left[\overbrace{\cos(n\pi)}^{(-1)^n}\right]$$

$$\int_0^1 x^3\cos(n\pi x)\,dx = \cancelto{0}{\left.\frac{x^3\sin(n\pi x)}{n\pi}\right|_0^1} - \frac{3}{n\pi}\int_0^1 x^2\sin(n\pi x)\,dx$$

$$= -\frac{3}{n\pi}\left[-x^2\frac{\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)\,dx\right]$$

$$= \frac{3}{(n\pi)^2}\left[\overbrace{\cos(n\pi)}^{(-1)^n}\right] - \frac{6}{(n\pi)^2}\int_0^1 x\cos(n\pi x)\,dx$$

$$= \frac{3(-1)^n}{(n\pi)^2} - \frac{6}{(n\pi)^2}\left[\cancelto{0}{\frac{x\sin(n\pi x)}{n\pi}\Big|_0^1} - \int_0^1\frac{\sin(n\pi x)}{n\pi}\,dx\right]$$

$$= \frac{3(-1)^n}{(n\pi)^2} + \frac{6}{(n\pi)^3}\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1$$

$$= \frac{3(-1)^n}{(n\pi)^2} + \frac{6}{(n\pi)^4}\left[1-(-1)^n\right]$$

$$\int_0^1 x^4 \cos(n\pi x)\,dx = \underbrace{\cancel{\frac{x^4 \sin(n\pi x)}{n\pi}\Bigg|_0^1}}_{0} - \frac{4}{n\pi}\int_0^1 \cancel{4}x^3 \frac{\sin(n\pi x)}{\cancel{n\pi}}\,dx$$

$$= \frac{-4}{n\pi}\left[-x^3\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \int_0^1 3x^2 \frac{\cos(n\pi x)}{n\pi}\,dx \right]$$

$$= \frac{4(-1)^n}{(n\pi)^2} - \frac{12}{(n\pi)^2}\int_0^1 x^2\cos(n\pi x)\,dx$$

$$= \frac{4(-1)^n}{(n\pi)^2} - \frac{12}{(n\pi)^2}\left[\frac{2}{(n\pi)^2}(-1)^n \right]$$

$$= \frac{4(-1)^n}{(n\pi)^2} - \frac{24(-1)^n}{(n\pi)^4}$$

$$\therefore\quad a_n \overset{n \ge 1}{=} 2\int_0^1 \varphi(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 x^2\cos(n\pi x)\,dx - 4\int_0^1 x^3\cos(n\pi x)\,dx + 2\int_0^1 x^4\cos(n\pi x)\,dx$$

$$= \frac{4(-1)^n}{(n\pi)^2} - 4\left[\frac{3(-1)^n}{(n\pi)^2} + \frac{6}{(n\pi)^4}\left[1-(-1)^n\right] \right] + 2\left[\frac{4(-1)^n}{(n\pi)^2} - \frac{24(-1)^n}{(n\pi)^4} \right]$$

$$= \frac{4(-1)^n - 12(-1)^n + 8(-1)^n}{(n\pi)^2} + \frac{-24\left(1-(-1)^n\right) - 48(-1)^n}{(n\pi)^4}$$

$$= \frac{24\left(-1-(-1)^n\right)}{(n\pi)^4}$$

$$= \begin{cases} \dfrac{-48}{(n\pi)^4} & \text{if } n = 2m \text{ is even,} \\[2ex] 0 & \text{if } n = 2m-1 \text{ is odd.} \end{cases}$$

etc.

Final Exam (2000 Summer)

$\mu = 127.1$

$\sigma = 49.0$

<u>Distribution of Scores</u>

174 –	A	4	(IIII)
146 – 173	B	1	(I)
120 – 145	C	3	(III)
100 – 119	D	4	(IIII)
0 – 99	F	4	(IIII)