Mathematics 325 | Final Exam Summer 2006 | Name: Dr. Grow (2 pts.)

On this examination, you may use the accompanying table of Fourier transforms and the statements of the Fourier series convergence theorems. In addition, you may find useful the following identities for the Laplacian of u in polar and spherical polar coordinates, respectively.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2}$$

1.(25 pts.) Solve $4t^3u_x + 3x^2u_t = 0$ subject to the initial condition $u(x,0) = x^6$ for all real x. Sketch the region in the $xt-$plane where the solution to this initial value problem is uniquely determined.

2.(25 pts.) Classify the type (hyperbolic, parabolic, or elliptic) of the linear second order partial differential equation $u_{xx} - 2u_{yy} + u_{xy} + u_x - u_y = 0$ and find, if possible, its general solution in the $xy-$plane.

3.(25 pts.) Solve $u_t - u_{xx} = 0$ in $-\infty < x < \infty,\ 0 < t < \infty$, subject to $u(x,0) = x^2$ for $-\infty < x < \infty$. (Note: You may find useful the following facts: $\int_{-\infty}^{\infty} pe^{-p^2}dp = 0,\ \ 2\int_{-\infty}^{\infty} p^2e^{-p^2}dp = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-p^2}dp.$)

4.(24 pts.) Use Fourier transform methods to derive a formula for the solution to

$$u_{tt} - c^2u_{xx} = f(x,t) \quad \text{in} \quad -\infty < x < \infty,\ -\infty < t < \infty,$$

subject to

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{if} \quad -\infty < x < \infty.$$

(Note: If you cannot solve the inhomogeneous problem then, for half the points, instead solve the homogeneous equation $u_{tt} - c^2u_{xx} = 0$ in the $xt-$plane subject to the initial conditions above.)

5.(25 pts.) (a) Show that the Fourier series of the function $f(x) = x(2-x)$ on $[0,1]$ with respect to the orthogonal system $\left\{\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\right\}_{n=0}^{\infty}$ on $[0,1]$ is $\sum_{n=0}^{\infty}\frac{32\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)}{(2n+1)^3\pi^3}$.

(b) Show that this Fourier series of f converges uniformly to f on $[0,1]$.

(c) Use these results to help evaluate the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$.

(d) Use these results to help evaluate the sum $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}$.

(The exam problems are continued on the back side of this page.)

6.(25 pts.) (a) Find a solution to $u_{tt} - u_{xx} = 0$ for $0 < x < 1$, $0 < t < \infty$, which satisfies $u(0,t) = 0 = u_x(1,t)$ for $t \geq 0$, and $u(x,0) = x(2-x)$, $u_t(x,0) = 0$ for $0 \leq x \leq 1$. (Hint: You may find the results of problem 5 useful here.)

(b) Show that there is at most one solution to the problem in part (a).

7.(24 pts.) The material in a spherical shell with inner radius 1 meter and outer radius 2 meters has a steady-state temperature distribution. The material is held at 100 degrees Centigrade on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies $u_r = -\kappa$ where κ is a positive constant.

(a) What is the temperature distribution function for this material?

(b) What are the hottest and coldest temperatures in the material?

(c) Is it possible to choose κ so that the temperature on the outer boundary is 20 degrees Centigrade? Please support your answers with reasons.

8.(25 pts.) Find a solution to

$$\frac{\partial^2 u}{\partial s^2} + \frac{1}{s}\frac{\partial u}{\partial s} + \frac{1}{s^2}\frac{\partial^2 u}{\partial t^2} = 0 \text{ for } 0 < s < 1,\ -\pi < t < \pi,$$

subject to the boundary conditions

$$u(s,\pi) = u(s,-\pi) \text{ and } u_t(s,\pi) = u_t(s,-\pi) \text{ for } 0 \leq s \leq 1$$

and

$$u(1,t) = 1 + \sin^3(t) \text{ for } -\pi \leq t \leq \pi.$$

(Please note that the solution u must be defined and continuous on $0 \leq s \leq 1$, $-\pi \leq t \leq \pi$.)

Bonus(10 pts.): Is there at most one solution to the above problem? Support your answer with reasons.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. **Uniform Convergence** The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. **L^2 Convergence** The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

Theorem 4. **Pointwise Convergence of Classical Fourier Series**

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

#1 $\quad 4t^3 u_x + 3x^2 u_t = 0 \quad , \quad u(x,0) = x^6 .$

The solution is constant along the characteristic curves $\dfrac{dt}{dx} = \dfrac{b(x,t)}{a(x,t)} = \dfrac{3x^2}{4t^3}$ 8pt to h

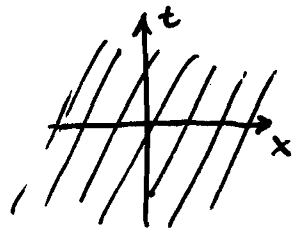
$\Rightarrow \quad 4t^3 dt = 3x^2 dx \quad \Rightarrow \quad t^4 = x^3 + c.$ Along such a characteristic curve,

the solution has value $u(x,t) = u(x, \sqrt[4]{x^3+c}) = u(0, \sqrt[4]{c}) = f(c)$. Thus

16 pts. to here. $u(x,t) = f(t^4 - x^3)$ is the general solution, where f is any C^1 function of a single real variable. To satisfy the auxiliary condition $u(x,0) = x^6 \quad (-\infty < x < \infty)$ we must choose f as follows: $\quad x^6 = u(x,0) = f(0^4 - x^3) = f(-x^3)$.

$\Rightarrow \quad f(z) = (-z)^2 = z^2$ for all real z. $\quad \therefore \quad \boxed{u(x,t) = (t^4 - x^3)^2}$ 24 pts. to here.

This solution is valid and unique in the entire xt-plane.



1 pt.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u$$

#2 $\quad \overbrace{u_{xx} + u_{xy} - 2u_{yy}} + u_x - u_y = 0 \quad \Rightarrow \quad B^2 - 4AC = 1^2 - 4(1)(-2) = 9 > 0.$

4 pts. to here The pde is $\boxed{\text{hyperbolic}}$. Let $\begin{cases} \xi = 2x - y \\ \eta = +x + y \end{cases}$. Then

8pts. to here.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 2\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \quad \Rightarrow \quad \frac{\partial}{\partial x} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \quad \Rightarrow \quad \frac{\partial}{\partial y} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} .$$

$$\therefore \quad \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} = \cancel{2\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} + 2\left(\cancel{-\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) = 3\frac{\partial}{\partial \eta}$$

and $$\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} - \left(-\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right) = 3\frac{\partial}{\partial \xi}$$

10 pts. to here.

Thus the pde $\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u+\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u=0$ is equivalent to

14 pts. to here

$\left(3\frac{\partial}{\partial \eta}\right)\left(3\frac{\partial}{\partial \xi}\right)u+3\frac{\partial u}{\partial \xi}=0$. Then $v=\frac{\partial u}{\partial \xi}$ produces $3\frac{\partial v}{\partial \eta}+v=0$,

18 pts. which has solution $v=c_1(\xi)e^{-\frac{1}{3}\eta}$. Then $\frac{\partial u}{\partial \xi}=c_1(\xi)e^{-\frac{1}{3}\eta}$ implies

24 pts. to here. $u=c_2(\xi)e^{-\frac{1}{3}\eta}+c_3(\eta)$. I.e. $\boxed{u(x,y)=f(2x-y)e^{-\frac{1}{3}(x+y)}+g(x+y)}$ 25 pts. to here

where f and g are arbitrary C^2 functions of a single real variable.

(#3) $u_t-u_{xx}=0$, $u(x,0)=x^2$ $(-\infty<x<\infty)$ 9 pts. to here

Let $p=\frac{y-x}{\sqrt{4t}}$.

Then $dp=\frac{dy}{\sqrt{4t}}$

5 pts. here. $u(x,t)=\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}\varphi(y)dy=\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}y^2dy$

14 pts. to here. $=\int_{-\infty}^{\infty}\frac{e^{-p^2}}{\sqrt{\pi}}(x+p\sqrt{4t})^2dp=\int_{-\infty}^{\infty}\frac{e^{-p^2}}{\sqrt{\pi}}(x^2+2px\sqrt{4t}+4tp^2)dp$

18 pts. to here. $=\frac{x^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-p^2}dp+\frac{2x}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-p^2}p\,dp+\frac{4t}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-p^2}p^2dp$

$=\frac{x^2}{\sqrt{\pi}}\cdot\sqrt{\pi}+\frac{2x}{\sqrt{\pi}}\cdot 0+\frac{4t}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=\boxed{x^2+2t}$ 23 pts. to here.

25 pts. to here. Check: $u(x,0)=x^2+2(0)\overset{\checkmark}{=}x^2$

$$\left.\begin{aligned}u_t&=2\\u_x&=2x\\u_{xx}&=2\end{aligned}\right\}\Rightarrow u_t-u_{xx}=2-2\overset{\checkmark}{=}0.$$

(#4) $u_{tt} - c^2 u_{xx} = f(x,t)$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ $(-\infty < x < \infty)$.

$$\mathcal{F}(u_{tt} - c^2 u_{xx})(\xi) = \mathcal{F}(f(x,t))(\xi)$$

3 pts. to here.

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + c^2\xi^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

2nd-order ODE in t with parameter ξ.

$$\left(W = \begin{vmatrix} \cos(c\xi t) & \sin(c\xi t) \\ -c\xi\sin(c\xi t) & c\xi\cos(c\xi t) \end{vmatrix} = c\xi \right)$$

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$$\mathcal{F}(u)(\xi) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t) + u_1(t)\cos(c\xi t) + u_2(t)\sin(c\xi t)$$

9

where $u_1(t) = \int_0^t -\frac{\hat{f}(\xi,\tau)\sin(c\xi\tau)}{c\xi}\,d\tau$ and $u_2(t) = \int_0^t \frac{\hat{f}(\xi,\tau)\cos(c\xi\tau)}{c\xi}\,d\tau$

12 pts to here.

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = c_1(\xi) \quad \text{and} \quad \mathcal{F}(\psi)(\xi) = \mathcal{F}(u_t)(\xi)\Big|_{t=0} = c\xi c_2(\xi).$$

$$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\cos(c\xi t) + \frac{1}{c\xi}\mathcal{F}(\psi)(\xi)\sin(c\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\sin(c\xi(t-\tau))\,d\tau$$

Using the identities $\cos(c\xi t) = \frac{1}{2}e^{ic\xi t} + \frac{1}{2}e^{-ic\xi t}$ and $\sin(c\xi t) = \frac{1}{2i}e^{ic\xi t} - \frac{1}{2i}e^{-ic\xi t}$ and the transform facts $\mathcal{F}(g(x-a))(\xi) = \hat{g}(\xi)e^{-i\xi a}$ and $\mathcal{F}\left(\int_{-\infty}^x g(s)\,ds\right) = \frac{\hat{g}(\xi)}{i\xi}$

15 pts. to here. 18 pts. to here.

yields

$$\mathcal{F}(u)(\xi) = \frac{1}{2}\mathcal{F}(\varphi)(\xi)e^{ic\xi t} + \frac{1}{2}\mathcal{F}(\varphi)(\xi)e^{-ic\xi t} + \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^x \psi(s)\,ds\right)(\xi)e^{ic\xi t} - \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^x \psi(s)\,ds\right)(\xi)e^{-ic\xi t}$$

$$+ \int_0^t \left[\frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^x f(s,\tau)\,ds\right)(\xi)e^{ic\xi(t-\tau)} - \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^x f(s,\tau)\,ds\right)(\xi)e^{-ic\xi(t-\tau)}\right]d\tau$$

$$= \frac{1}{2}\mathcal{F}(\varphi(x+ct))(\xi) + \frac{1}{2}\mathcal{F}(\varphi(x-ct))(\xi) + \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x+ct}\psi(s)\,ds\right)(\xi) - \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x-ct}\psi(s)\,ds\right)(\xi)$$

$$+ \int_0^t \left[\frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x+c(t-\tau)} f(s,\tau)\,ds\right)(\xi) - \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x-c(t-\tau)} f(s,\tau)\,ds\right)(\xi)\right]d\tau$$

$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right]\right)(\xi) + \mathcal{F}\left(\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds\right)(\xi)$$

$$+ \int_0^t \mathcal{F}\left(\frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\right)(\xi)\,d\tau$$

21 pts. to here.

Interchanging the order of integration in the last term and using linearity of the Fourier transform gives

$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau\right)(\xi).$$

Applying the inversion formula leads to

24 p to here.

$$\boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau}.$$

#5 (a) +7 $f(x) = x(2-x)$ on $[0,1]$ $\sim \sum_{n=0}^{\infty} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$ where

$$b_n = \frac{\langle f, \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\rangle}{\langle \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right), \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\rangle} = 2\int_0^1 \overbrace{x(2-x)}^{v}\overbrace{\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)dx}^{dv} = \cancel{\left.\frac{2x(2-x)\left(-\cos\left(\left(\frac{2n+1}{2}\right)\pi x\right)\right)}{\left(\frac{2n+1}{2}\right)\pi}\right|_0^1}$$

$$+\frac{2^2}{\pi}\int_0^1 \frac{2}{2n+1}\overbrace{(2-2x)}^{v}\overbrace{\cos\left(\left(\frac{2n+1}{2}\right)\pi x\right)dx}^{dv} = \cancel{\left.\frac{2^2}{\pi(2n+1)}(2-2x)\frac{2}{(2n+1)\pi}\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\right|_0^1} - \frac{4\cdot 2}{\pi^2(2n+1)^2}\int_0^1 \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$$

$$= \frac{8\cdot 2}{\pi^2(2n+1)^2}\left[\frac{-2}{(2n+1)\pi}\cos\left(\left(\frac{2n+1}{2}\right)\pi x\right)\right]\Bigg|_0^1 = \frac{32}{\pi^3(2n+1)^3}.$$

$$\therefore \quad x(2-x) \sim \sum_{n=0}^{\infty} \frac{32\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)}{\pi^3(2n+1)^3} \quad \text{on } [0,1].$$

#5 (b) +6

$$\left.\begin{aligned} f(x) &= x(2-x) = 2x - x^2 \\ f'(x) &= 2-2x \\ f''(x) &= -2 \end{aligned}\right\} \text{ continuous functions on } 0 \le x \le 1.$$

The hermitian boundary $\underbrace{\text{conditions}}_{\text{for } T = -\frac{d^2}{dx^2}}$ that generate the eigenfunctions $\left\{\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\right\}_n$ are $X(0) = 0$ and $X'(1) = 0$. Note that $f(0) = 0$ and $f'(1) = 2-2(1) = 0$ so f satisfies the B.C.s. Therefore, Theorem 2 implies that the Fourier series $\displaystyle\sum_{n=0}^{\infty} \frac{32\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)}{\pi^3(2n+1)^3}$ converges uniformly to $f(x) = (2-x)x$ on $[0,1]$.

+6 (c) Evaluate $\displaystyle x(2-x) = \sum_{n=0}^{\infty} \frac{32\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)}{\pi^3(2n+1)^3}$ at $x = 1$:

Then $\displaystyle 1 = 1(2-1) = \sum_{n=0}^{\infty} \frac{32\sin\left((2n+1)\frac{\pi}{2}\right)}{\pi^3(2n+1)^3} = \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$

so $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \boxed{\frac{\pi^3}{32}}$.

+6 (d) By Parseval's identity; $\displaystyle\sum_{n=0}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$,

we have $\displaystyle\sum_{n=0}^{\infty} \left|\frac{32}{\pi^3(2n+1)^3}\right|^2 \int_0^1 \sin^2\left(\left(\frac{2n+1}{2}\right)\pi x\right)dx = \int_0^1 |x(2-x)|^2 dx$.

Evaluating integrals gives $\displaystyle\sum_{n=0}^{\infty} \frac{32^2}{\pi^6(2n+1)^6}\cdot\frac{1}{2} = \frac{8}{15}$

so $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{8}{15}\cdot\frac{2\pi^6}{32^2} = \boxed{\frac{\pi^6}{960}}$.

#6 (a) $u_{tt} - u_{xx} \overset{①}{=} 0$, $u(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(1,t)$ if $t \geq 0$ and $u(x,0) \overset{⑤}{=} x(2-x)$, $u_t(x,0) \overset{④}{=} 0$ if $0 \leq x \leq 1$

We seek nontrivial solutions to ①-②-③-④ of the form $u(x,t) = X(x)T(t)$. This

5 pts. to here. leads to $\begin{cases} \boxed{X'' + \lambda X = 0, \ X(0) = 0, \ X'(1) = 0} \leftarrow \text{Eigenvalue Problem} \\ T'' + \lambda T = 0, \ T'(0) = 0. \end{cases}$

Since the operator $L = -\frac{d^2}{dx^2}$ is hermitian on $V = \{ f \in C^2[0,1] : f(0) = 0 = f'(1) \}$, all the eigenvalues are real. In fact, all the eigenvalues are nonnegative. To see this, let λ be an eigenvalue and $0 \neq X$ in V be a corresponding eigenfunction. Then

$$\lambda \langle X, X \rangle = \langle \lambda X, X \rangle = \langle -X'', X \rangle = -\int_0^1 \underbrace{\overline{X(x)}}_{v} \underbrace{X''(x)dx}_{dV} = -\overline{X(x)}X'(x)\Big|_0^1 + \int_0^1 |X'(x)|^2$$

7 But $\overline{X(0)} = 0 = X'(1)$ so $\lambda \overbrace{\langle X, X \rangle}^{\text{positive}} = \langle X', X' \rangle \geq 0 \Rightarrow \lambda \geq 0.$

<u>Case 1</u>: $\lambda > 0$, say $\lambda = \alpha^2$ where $\alpha > 0$. Then $X'' + \alpha^2 X = 0$ has general solution $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. Then $X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$ so

$0 = X(0) = c_1$ and $0 = X'(1) = \alpha c_2 \cos(\alpha) \Rightarrow \alpha = \alpha_n = \left(\frac{2n+1}{2}\right)n\pi$

$(n = 0, 1, 2, \ldots)$. Thus $\lambda_n = \alpha_n^2 = \left(\frac{2n+1}{2}\right)^2 n^2 \pi^2$ and $X_n(x) = \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$

10 are the eigenvalues and eigenfunctions, respectively.

<u>Case 2</u>: $\lambda = 0$. Then $X'' = 0$ so $X'(x) = c_1$ and $X''(x) = c_1 x + c_2$.

$0 = X(0) = c_2$ and $0 = X'(1) = c_1$ so the only solution is trivial.

11 That is, $\lambda = 0$ is not an eigenvalue.

$T_n''(t) + \lambda_n T_n(t) = 0, \ T_n'(0) = 0 \Rightarrow T_n''(t) + \left(\frac{2n+1}{2}\right)^2 n^2 \pi^2 T_n(t) = 0, \ T_n'(0) =$

13 $\Rightarrow T_n(t) = \cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)$.

By the superposition principle $u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)$ is a

15 formal solution to ①-②-③-④ for any constants $b_0, b_1, b_2, \ldots$ To satisfy ⑤ :

$x(2-x) = u(x,0) = \sum_{n=0}^{\infty} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$ for $0 \le x \le 1$, we should choose

$b_n = \frac{32}{\pi^3(2n+1)^3}$ for $n = 0, 1, 2, \ldots$ by problem 5. Therefore

18

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{32 \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)}{\pi^3 (2n+1)^3}}.$$

(b) Suppose $u = v(x,t)$ were another solution to the problem in part (a). Then

$w(x,t) = u(x,t) - v(x,t)$ would solve ①-②-③-④ and ⑤': $u(x,0) = 0$ for $0 \le x \le 1$.

2 pts. to here.

Consider the energy $E(t) = \int_0^1 \left[\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t)\right] dx$ of w at time t. Then

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t)\right] dx = \int_0^1 \left[w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t)\right] dx$$

$$= \int_0^1 \left[w_t(x,t) w_{xx}(x,t) + w_x(x,t) w_{xt}(x,t)\right] dx = \int_0^1 \frac{\partial}{\partial x}\left(w_t(x,t) w_x(x,t)\right) dx = w_t(x,t) w_x(x,t) \Big|_{x:}$$

5 pts. to here

But $w_x(1,t) = 0$ and $w(0,t) = 0$ implies $w_t(0,t) = 0$ for $t \ge 0$ so $\frac{dE}{dt} = 0$.

Thus E is constant: $E(t) = E(0) = \int_0^1 \left[\frac{1}{2} w_t^2(x,0) + \frac{1}{2} w_x^2(x,0)\right] dx = 0$ by ④ and ⑤'.

$\therefore$ the first vanishing theorem implies $w_t(x,t) = 0 = w_x(x,t)$ for all $0 \le x \le 1$ and $t \ge$

so $w(x,t) =$ constant. But then either ② or ⑤ implies $w(x,t) = 0$; that is,

7 pts. to here

$u(x,t) = v(x,t)$ for all $0 \le x \le 1$, $0 \le t$, proving uniqueness.

#7 (a) The steady-state temperature $u = u(r,\theta,\varphi)$ satisfies $\nabla^2 u = 0$ in $1 < r < 2$ and the boundary conditions $u(1,\theta,\varphi) = 100$ and $u_r(2,\theta,\varphi) = -K$. By radial symmetry of the pde, region, and boundary conditions we may assume

3 pts. to here

that $u = u(r)$, independent of θ and φ. Then $\nabla^2 u = 0$ becomes

6 $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=0 \Rightarrow r^2\frac{\partial u}{\partial r}=c_1$. Then $-K=u_r(2)=\frac{c_1}{2^2}$

implies $4(-K)=c_1$ so $r^2\frac{\partial u}{\partial r}=-4K \Rightarrow \frac{\partial u}{\partial r}=\frac{-4K}{r^2}$ and

7

12 $u(r)=c_2+\frac{4K}{r}$. Then $100=u(1)=c_2+4K \Rightarrow c_2=100-4K$

so $\boxed{u(r)=100-4K+\frac{4K}{r}}$

15 pts. to here.

6 (b) The hottest temperature in the material occurs on the inner boundary: $\boxed{u(1)=100}$

The coldest temperature in the material occurs on the outer boundary: $\boxed{u(2)=100-2K}$.

3 (c) Yes, it is possible to choose K so $u(2)=20$. In fact

$$20=100-2K \Rightarrow 2K=80 \Rightarrow \boxed{K=40}.$$

#8 Method 1: With $r=s$ and $\theta=t$, the problem is equivalent to

$\nabla^2 u=0$ in the disk $0\le r<1$ with $u(1;\theta)=\underbrace{1+\sin^3(\theta)}_{h(\theta)}$ for $-\pi\le\theta\le\pi$

(The periodic boundary conditions $u(r;\pi)=u(r;-\pi)$ and $u_\theta(r;\pi)=u_\theta(r;-\pi)$ for $0\le r\le 1$ are natural in polar coordinates since $(r;\pi)$ and $(r;-\pi)$ refer to the same point in the disk.) Therefore $u(r;\theta)=\sum_{n=-\infty}^{\infty}\hat{h}(n)e^{in\theta}r^{|n|}$ is the solution where $\hat{h}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(\theta)e^{-in\theta}d\theta$. But $h(\theta)=1+\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3=$

$1+\frac{1}{-8i}\left(e^{3i\theta}-3e^{2i\theta}e^{-i\theta}+3e^{i\theta}e^{-2i\theta}-e^{-3i\theta}\right)=1-\frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)+\frac{3}{4}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$

$=1-\frac{1}{4}\sin(3\theta)+\frac{3}{4}\sin(\theta)$. Therefore $\hat{h}(0)=1$, $\hat{h}(1)=\frac{3}{8i}$, $\hat{h}(-1)=-\frac{3}{8i}$, $\hat{h}(3)=-\frac{1}{8i}$

and $\hat{h}(-3)=\frac{1}{8i}$. Also $\hat{h}(n)=0$ for all other n.

$$\therefore\ u(r;\theta) = \hat{h}(0) + r\left(\hat{h}(1)e^{i\theta} + \hat{h}(-1)e^{-i\theta}\right) + r^3\left(\hat{h}(3)e^{3i\theta} + \hat{h}(-3)e^{-3i\theta}\right)$$

$$= 1 + r\left(\frac{3e^{i\theta}}{8i} - \frac{3e^{-i\theta}}{8i}\right) + r^3\left(-\frac{1}{8i}e^{3i\theta} + \frac{1}{8i}e^{-3i\theta}\right)$$

$$= 1 + \frac{3r}{4}\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) - \frac{r^3}{4}\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)$$

$$= 1 + \frac{3}{4}r\sin(\theta) - \frac{r^3}{4}\sin(3\theta)$$

In terms of the original variables s and t,

$$\boxed{u(s,t) = 1 + \frac{3s\sin(t)}{4} - \frac{s^3\sin(3t)}{4}}$$

<u>Bonus</u>: There is at most one solution to this problem by the uniqueness theorem for harmonic functions in a bounded region satisfying Dirichlet boundary conditions. (This is an immediate consequence of the maximum/minimum principle for harmonic functions.)