Mathematics 325
Homework 14

Due Date: ____________________

Name: __________________________

(a) Find a solution to

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1,\ 0 < t < \infty,$$

subject to

$$u_x(0,t) = 0 = u_x(1,t) \quad \text{for} \quad t \geq 0,$$

and

$$u(x,0) = \cos^2(\pi x), \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq 1.$$

(b) Use the energy method to show that there is only one solution to the problem in part (a).

HW 14 : (a) $u(x,t) = X(x)T(t)$ in the homogeneous portion of the problem leads to
$$\begin{cases} X''(x) + \lambda X(x) = 0, & X'(0) = 0 = X'(1), \\ T''(t) + \lambda T(t) = 0, & T'(0) = 0. \end{cases}$$
10%

20% The eigenvalues are $\lambda_n = (n\pi)^2$ and the eigenfunctions are $X_n(x) = \cos(n\pi x)$ $(n = 0, 1, 2, \ldots)$

30% The solution to the $t$-problem is $T_n(t) = \cos(n\pi t)$ $(n = 0, 1, 2, \ldots)$ Hence

40%
$$u(x,t) = \sum_{n=0}^{N} a_n \cos(n\pi x)\cos(n\pi t)$$

solves the homogeneous portion of the problem for any $N \geq 1$ and any constants $a_0, \ldots, a_N$

$$\frac{1}{2} + \frac{1}{2}\cos(2\pi x) = \cos^2(\pi x) \overset{\text{Want!}}{=} u(x,0) = \sum_{n=0}^{N} a_n \cos(n\pi x) \text{ for } 0 \leq x \leq 1 \Rightarrow a_0 = \frac{1}{2}, a_2 = \frac{1}{2}, \text{ and}$$

50% all other $a_n = 0$. $\therefore$ $\boxed{u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)\cos(2\pi t)}$

(b) Let $v = v(x,t)$ be any other solution to the problem in (a) and consider the

15% energy function $E(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]\,dx$ of the difference $w(x,t) = u(x,t) - v(x,t)$.

Note that $w$ solves $w_{tt} - w_{xx} \overset{①}{=} 0$ in $0 < x < 1$, $0 < t < \infty$, $w_x(0,t) \overset{②}{=} 0 \overset{③}{=} w_x(1,t)$ for $t \geq 0$,

$w(x,0) \overset{④}{=} 0 \overset{⑤}{=} w_t(x,0)$ for $0 \leq x \leq 1$. $\dfrac{dE}{dt} = \dfrac{1}{2}\displaystyle\int_0^1 \frac{\partial}{\partial t}[w_t^2(x,t) + w_x^2(x,t)]\,dx =$

$$\int_0^1 [w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{xt}(x,t)]\,dx \overset{①}{=} \int_0^1 [w_t(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t)]\,dx$$

36% $$= \int_0^1 \frac{\partial}{\partial x}[w_t(x,t)\,w_x(x,t)]\,dx = w_t(1,t)\overbrace{w_x(1,t)}^{0} - w_t(0,t)\overbrace{w_x(0,t)}^{0} \overset{②\text{-}③}{=} 0.$$ Therefore,

40% for all $t \geq 0$, $E(t) = E(0) = \frac{1}{2}\int_0^1 [w_t^2(x,0) + w_x^2(x,0)]\,dx = 0$. By the vanishing theorem, it follows that $\frac{1}{2}[w_t^2(x,t) + w_x^2(x,t)] = 0$ for all $0 \leq x \leq 1$ and all $t \geq 0$.

Consequently $w_t(x,t) = w_x(x,t) = 0$ for all $0 \leq x \leq 1$ and all $0 \leq t$. It follows that $w(x,t) = $ constant for all $0 \leq x \leq 1$, $0 \leq t$. But ④ implies this constant is zero.

50% I.e. $\boxed{u(x,t) = v(x,t) \text{ for all } 0 \leq x \leq 1 \text{ and all } t \geq 0.}$