Mathematics 325
Homework 15

Due Date: ________________

Name: ____________________

A thin rod of unit length has its lateral surface insulated against the flow of heat. The material comprising the rod has thermal conductivity 5.2, specific heat 4.0, and density 1.3. The left end of the rod is insulated:

$$u_x(0,t) = 0 \quad \text{for } t \geq 0,$$

its right end radiates freely into air of constant temperature zero:

$$u_x(1,t) + 0.5u(1,t) = 0 \quad \text{for } t \geq 0,$$

and the initial temperature distribution in the rod is uniform:

$$u(x,0) = 100 \quad \text{for } 0 \leq x \leq 1.$$

15% (a) Use separation of variables and deduce the corresponding eigenvalue problem.

15% (b) Find the condition(s) satisfied by the eigenvalues of this problem. Please show the details for the calculation in each "case".

15% (c) Show that there exists an infinite sequence of eigenvalues for this problem and numerically approximate to eight decimal place accuracy the first three eigenvalues.

10% (d) Write the corresponding eigenfunctions and the "asymptotic behavior" of the eigenvalues.

15% (e) Find a series expression for the temperature $u = u(x,t)$ at any point x in the rod at any subsequent time t. Although you do not have to evaluate them, be sure to give explicit formulas for the coefficients in your series expression for the temperature function.

15% (f) Approximate to eight decimal place accuracy the first three coefficients of your series expression.

15% (g) Truncate your series expression for the temperature to the first three terms and use this to estimate the temperature of the rod at position $x = 0.5$ and time $t = 1.0$. How accurate is your answer? How do you know this?

(a) Let $u(x,t)$ denote the temperature of the rod at position x in $[0,1]$ and time $t \geq 0$. Then u is a solution of the diffusion equation

$$u_t - k u_{xx} = 0 \quad \text{in } 0<x<1,\ 0<t<\infty,$$

5% where $k = \dfrac{K}{\sigma\rho} = \dfrac{5.2}{(4.0)(1.3)} = 1$. Thus u solves

$$\begin{cases} u_t - u_{xx} \overset{①}{=} 0 & \text{in } 0<x<1,\ 0<t<\infty, \\ u_x(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(1,t) + \frac{1}{2}u(1,t) & \text{for } t \geq 0, \\ u(x,0) \overset{④}{=} 100 & \text{for } 0 \leq x \leq 1. \end{cases}$$

We seek nontrivial solutions of ①-②-③ of the form $u(x,t) = X(x)T(t)$.
Thus $X(x)T'(t) - X''(x)T(t) = 0 \Rightarrow -\dfrac{X''(x)}{X(x)} = \dfrac{-T'(t)}{T(t)} = \text{constant} = \lambda.$

$X'(0)T(t) = 0 = X'(1)T(t) + \frac{1}{2}X(1)T(t)$ for $t \geq 0$ so

10% $$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(1) + \frac{1}{2}X(1),} \\ T'(t) + \lambda T(t) = 0. \end{cases}$$ Eigenvalue Problem

(b) <u>Case 1</u>: $\lambda > 0$, say $\lambda = \alpha^2$ where $\alpha > 0$.

$$X'' + \alpha^2 X = 0 \Rightarrow X(x) = c_1\cos(\alpha x) + \cancel{c_2}^{0}\sin(\alpha x)$$

$$X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha \cancel{c_2}^{0}\cos(\alpha x)$$

$$0 = X'(0) = \alpha c_2 \Rightarrow c_2 = 0.$$

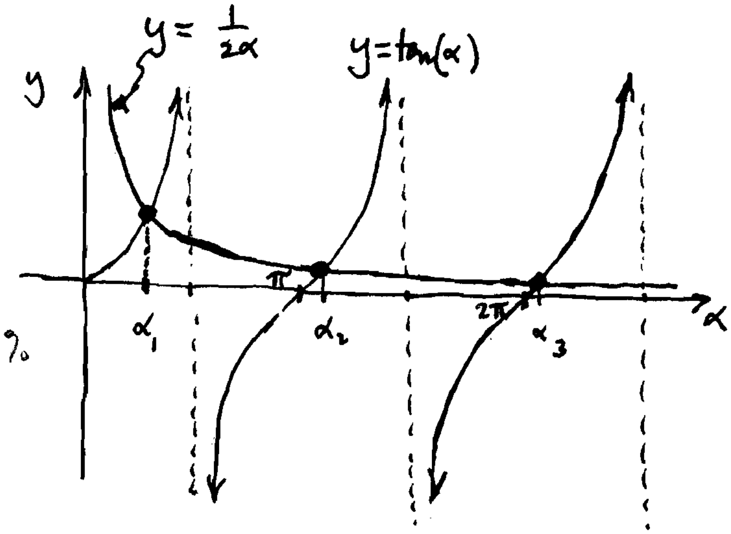
Eigenvalue condition.

$$0 = X'(1) + \tfrac{1}{2}X(1) = \underbrace{-\alpha c_1 \sin(\alpha) + \frac{c_1}{2}\cos(\alpha)}_{c_1\left(-\alpha\sin(\alpha) + \frac{1}{2}\cos(\alpha)\right)} \Rightarrow \boxed{\tan(\alpha) = \frac{1}{2\alpha}} \quad 5\%$$

nonzero, so solution isn't trivial.

(c)

5%



Use a calculator to find:

$\alpha_1 \doteq 0.653271187094$

10% $\Rightarrow \boxed{\lambda_1 = \alpha_1^2 \doteq 0.42676324}$

$\alpha_2 \doteq 3.29231002128$

$\Rightarrow \boxed{\lambda_2 = \alpha_2^2 \doteq 10.83930528}$

$\alpha_3 \doteq 6.36162039207$

$\Rightarrow \boxed{\lambda_3 = \alpha_3^2 \doteq 40.47021401}$

It is clear that there is an infinite sequence of solutions to $\tan(\alpha) = \frac{1}{2\alpha}$, say $0 < \alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with the properties $\alpha_n \in \left((n-1)\pi, (n-\tfrac{1}{2})\pi\right)$ and $\lim_{n\to\infty} \left[\alpha_n - (n-1)\pi\right] = 0$. The corresponding eigenfunctions

5%

(d) are (up to a constant factor) $\boxed{X_n(x) = \cos(\alpha_n x)}$ with eigenvalues $\boxed{\lambda_n = \alpha_n^2}$ for $n = 1, 2, 3, \ldots$ Note that $\boxed{\lim_{n\to\infty}\left[\lambda_n - (n-1)^2\pi^2\right] = 0.}$ 5%

(b) 5% Case 2: $\lambda = 0$. $X'' = 0 \Rightarrow X'(x) = c_1$ and $X(x) = c_1 x + c_2$

(cont.) $0 = X'(0) = c_1$ and $0 = X'(1) + \frac{1}{2}X(1) = c_1 + \frac{1}{2}(c_1 + c_2) = \frac{1}{2}c_2$.

No nontrivial solutions exist in this case.

5% Case 3: $\lambda < 0$, say $\lambda = -\beta^2$ where $\beta > 0$.

$X'' - \beta^2 X = 0 \Rightarrow X(x) = c_1\cosh(\beta x) + \cancel{c_2}^{\,0}\sinh(\beta x)$

$X'(x) = \beta c_1 \sinh(\beta x) + \beta \cancel{c_2}_{\,0}\cosh(\beta x)$

$0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$.

$0 = X'(1) + \frac{1}{2}X(1) = \beta c_1 \sinh(\beta) + \frac{1}{2}c_1\cosh(\beta) = c_1\left(\overbrace{\beta\sinh(\beta) + \tfrac{1}{2}\cosh(\beta)}^{\text{positive when } \beta>0}\right)$

$\Rightarrow c_1 = 0$.

No nontrivial solutions exist in this case.

(e) For any $n \geq 1$, the solution to $T_n'(t) + \lambda_n T_n(t) = 0$ is

$T_n(t) = e^{-\lambda_n t} = e^{-\alpha_n^2 t}$, up to a constant multiple. The formal series solution to ①-②-③ is $u(x,t) = \sum_{n=1}^{\infty} c_n \cos(\alpha_n x) e^{-\alpha_n^2 t}$, where $c_1, c_2, c_3, \ldots$ are arbitrary constants. In order to satisfy ④, we must choose the constants so that

$$100 = u(x,0) = \sum_{n=1}^{\infty} c_n \overbrace{\cos(\alpha_n x)}^{X_n(x)} \quad \text{for } 0 \leq x \leq 1.$$

Since $\{X_n\}_{n=1}^{\infty} = \{\cos(\alpha_n x)\}_{n=1}^{\infty}$ is an orthogonal system on $(0,1)$ (see the claim and its proof on the last page of this homework solution), we may take c_n to be the n^{th} Fourier coefficient of $f(x) = 100$ with respect to the orthogonal system $\{X_n\}_{n=1}^{\infty}$:

$$c_n = \frac{\langle 100, \cos(\alpha_n x)\rangle}{\langle \cos(\alpha_n x), \cos(\alpha_n x)\rangle} = \frac{\int_0^1 100\cos(\alpha_n x)\,dx}{\int_0^1 \cos^2(\alpha_n x)\,dx} = \frac{\left.\frac{100\sin(\alpha_n x)}{\alpha_n}\right|_{x=0}^{1}}{\int_0^1 \frac{1+\cos(2\alpha_n x)}{2}\,dx}$$

$$\boxed{c_n =} \frac{\frac{100\sin(\alpha_n)}{\alpha_n}}{\left[\frac{1}{2}x + \frac{1}{4\alpha_n}\sin(2\alpha_n x)\right]\Big|_{x=0}^{1}} = \frac{\frac{100\sin(\alpha_n)}{\alpha_n}}{\frac{1}{2} + \frac{\sin(2\alpha_n)}{4\alpha_n}} = \boxed{\frac{400\sin(\alpha_n)}{2\alpha_n + \sin(2\alpha_n)}}$$

Therefore the solution to ①-②-③-④ is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{400\sin(\alpha_n)\cos(\alpha_n x)e^{-\alpha_n^2 t}}{2\alpha_n + \sin(2\alpha_n)}}.$$

15%

(f) The first three coefficients of the series expression for the solution obtained in part (e) are:

$$c_1 = \frac{400\sin(\alpha_1)}{2\alpha_1 + \sin(2\alpha_1)} \doteq \frac{400\sin(0.65327187094)}{2(0.65327187094) + \sin((2)(0.65327187094))}$$

5% $\Rightarrow \boxed{c_1 \doteq 107.012813694}$

$$c_2 = \frac{400\sin(\alpha_2)}{2\alpha_2 + \sin(2\alpha_2)} \doteq \frac{400\sin(3.29231002128)}{2(3.29231002128) + \sin(2(3.29231002128))}$$

5% $\Rightarrow \boxed{c_2 \doteq -8.72758410879}$

$$c_3 = \frac{400\sin(\alpha_3)}{2\alpha_3 + \sin(2\alpha_3)} \doteq \frac{400\sin(6.36162039207)}{2(6.36162039207) + \sin(2(6.36162039207))}$$

5% $\boxed{c_3 \doteq 2.43347580818}$.

(g) $$u(x,t) \doteq \sum_{k=1}^{3} c_k \cos(\alpha_k x) e^{-\alpha_k^2 t} = u_3(x,t).$$

$$u(\tfrac{1}{2},1) \doteq c_1\cos(\alpha_1(\tfrac{1}{2}))e^{-\alpha_1^2} + c_2\cos(\alpha_2(\tfrac{1}{2}))e^{-\alpha_2^2} + c_3\cos(\alpha_3(\tfrac{1}{2}))e^{-\alpha_3^2}$$

$$\doteq 66.1459365805 + 0.0000128874129845 + -6\times 10^{-18}$$

10% $$\doteq \boxed{66.1459494679}$$

The error in using the sum of the first three terms to approximate $u(1/2, 1)$ is

$$\left| u\left(\tfrac{1}{2},1\right) - u_3\left(\tfrac{1}{2},1\right) \right| = \left| \sum_{n=4}^{\infty} c_n \cos\left(\alpha_n\left(\tfrac{1}{2}\right)\right) e^{-\alpha_n^2} \right|$$

$$\leq \sum_{n=4}^{\infty} |c_n| e^{-\alpha_n^2}. \qquad \left(\text{since } \left|\cos\left(\alpha_n\left(\tfrac{1}{2}\right)\right)\right| \leq 1\right)$$

But $$|c_n| = \left| \frac{400 \sin(\alpha_n)}{2\alpha_n + \sin(2\alpha_n)} \right| \leq \frac{400}{2\alpha_n - 1} \leq \frac{400}{2(n-1)\pi - 1} \overset{\text{for } n \geq 4.}{\leq} 22.5.$$

$$\therefore \left| u\left(\tfrac{1}{2},1\right) - u_3\left(\tfrac{1}{2},1\right) \right| \leq 22.5 \sum_{n=4}^{\infty} e^{-\alpha_n^2}$$

$$< 22.5 \sum_{n=4}^{\infty} \left(e^{-\alpha_4}\right)^n$$

$$(\alpha_4 \doteq 9.47748570542)$$

$$= 22.5 \, \frac{\left(e^{-\alpha_4}\right)^4}{1 - e^{-\alpha_4}}$$

$$\doteq \boxed{7.73 \times 10^{-16}}$$

5%

Claim: The eigenfunctions $X_n(x) = \cos(\alpha_n x)$ $(n = 1, 2, 3, \ldots)$ form an orthogonal system on $(0,1)$.

Reason: Note that for each $n = 1, 2, 3, \ldots$ we have

$$X_n''(x) + \alpha_n^2 X_n(x) = 0 \quad \text{for } 0 \le x \le 1, \quad X_n'(0) = 0, \text{ and}$$

$$X_n'(1) + \tfrac{1}{2} X_n(1) = 0. \quad \text{Therefore}$$

$$\langle X_m, X_n \rangle = \int_0^1 X_m(x) \overline{X_n(x)}\, dx$$

$$= \int_0^1 X_m(x) \left(- X_n''(x) \cdot \frac{1}{\alpha_n^2} \right) dx$$

integrate by parts twice

$$= -\frac{1}{\alpha_n^2} \Big(X_m(x) X_n'(x) - X_m'(x) X_n(x) \Big) \Big|_{x=0}^{1} - \frac{1}{\alpha_n^2} \int_0^1 X_m''(x) X_n(x) d$$

$$= -\frac{1}{\alpha_n^2} \Big(X_m(1) \overset{-\frac{1}{2}X_n(1)}{\cancel{X_n'(1)}} - \overset{-\frac{1}{2}X_m(1)}{\cancel{X_m'(1)}} X_n(1) - X_m(0) \cancelto{0}{X_n'(0)} + \cancelto{0}{X_m'(0)} X_n(0) \Big)$$

$$- \frac{1}{\alpha_n^2} \int_0^1 \left(-\alpha_m^2 X_m(x) \right) X_n(x)\, dx$$

$$= \frac{\alpha_m^2}{\alpha_n^2} \langle X_m, X_n \rangle .$$

Therefore $\left(1 - \dfrac{\alpha_m^2}{\alpha_n^2}\right) \langle X_m, X_n \rangle = 0$, so if $m \ne n$ (and hence $\alpha_m^2 \ne \alpha_n^2$) we must have $\langle X_m, X_n \rangle = 0$.