Mathematics 325
Homework Assignment 1

Due Date:______________________

Name:___________________________

Work exercise 3 on page 5 of Strauss.

3. **For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.**
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1 + u} = 0$

#3 (a) If $\mathcal{L}(u) = u_t - u_{xx}$ then the PDE may be expressed as $\mathcal{L}(u) = -1$

$$\mathcal{L}(u+v) = (u+v)_t - (u+v)_{xx} = u_t + v_t - u_{xx} - v_{xx} = u_t - u_{xx} + v_t - v_{xx} = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(ku) = (ku)_{tt} - (ku)_{xx} = ku_{tt} - ku_{xx} = k(u_{tt} - u_{xx}) = k\mathcal{L}(u)$$

Therefore the PDE is $\boxed{\text{linear, inhomogeneous, and of second order}}$.

(b) If $\mathcal{L}(u) = u_t - u_{xx} + xu$ then the PDE may be expressed as $\mathcal{L}(u) = 0$

$$\mathcal{L}(u+v) = (u+v)_t - (u+v)_{xx} + x(u+v) = u_t - u_{xx} + xu + v_t - v_{xx} + vx = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(ku) = (ku)_t - (ku)_{xx} + x(ku) = k(u_t - u_{xx} + xu) = k\mathcal{L}(u).$$

Therefore the PDE is $\boxed{\text{linear, homogeneous, and of second order}}$.

(c) If $\mathcal{L}(u) = u_t - u_{xxt} + uu_x$ then the PDE may be expressed as $\mathcal{L}(u) = 0$

$$\mathcal{L}(ku) = (ku)_t - (ku)_{xxt} + (ku)(ku)_x = ku_t - ku_{xxt} + k^2uu_x$$

$$k\mathcal{L}(u) = k(u_t - u_{xxt} + uu_x) = ku_t - ku_{xxt} + kuu_x$$

If $k \neq 0, 1$ and $uu_x \neq 0$ then $k^2 uu_x \neq kuu_x$ so $\mathcal{L}(ku) \neq k\mathcal{L}(u)$.

Therefore the PDE is $\boxed{\text{nonlinear and of third order}}$.

(d) If $\mathcal{L}(u) = u_{tt} - u_{xx}$ then the PDE may be expressed as $\mathcal{L}(u) = -x^2$.

$$\mathcal{L}(u+v) = (u+v)_{tt} - (u+v)_{xx} = u_{tt} - u_{xx} + v_{tt} - v_{xx} = \mathcal{L}(u) + \mathcal{L}(v).$$

$$\mathcal{L}(ku) = (ku)_{tt} - (ku)_{xx} = ku_{tt} - ku_{xx} = k\mathcal{L}(u).$$

Therefore the PDE is $\boxed{\text{linear, inhomogeneous, and of second order}}$.

(e) If $\mathcal{L}(u) = iu_t - u_{xx} + \frac{1}{x}u$ then the PDE may be expressed as $\mathcal{L}(u) =$

$$\mathcal{L}(u+v) = i(u+v)_t - (u+v)_{xx} + \frac{1}{x}(u+v) = iu_t - u_{xx} + \frac{1}{x}u + iv_t - v_{xx} + \frac{1}{x}v = \mathcal{L}(u) + \mathcal{L}(v$$

$$\mathcal{L}(ku) = i(ku)_t - (ku)_{xx} + \frac{1}{x}(ku) = k\left(iu_t - u_{xx} + \frac{1}{x}u\right) = k\mathcal{L}(u).$$

Therefore the PDE is $\boxed{\text{linear, homogeneous, and of second order}}$.

(f) If $\mathcal{L}(u) = \dfrac{u_x}{\sqrt{1+u_x^2}} + \dfrac{u_y}{\sqrt{1+u_y^2}}$ then the PDE may be expressed as $\mathcal{L}(u) = 0$

$$\mathcal{L}(ku) = \frac{(ku)_x}{\sqrt{1+(ku)_x^2}} + \frac{(ku)_y}{\sqrt{1+(ku)_y^2}} = k\left(\frac{u_x}{\sqrt{1+k^2u_x^2}} + \frac{u_y}{\sqrt{1+k^2u_y^2}}\right)$$

$$k\mathcal{L}(u) = k\left(\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_y}{\sqrt{1+u_y^2}}\right).$$

Notice that if $k^2 > 1$ and $u_x > 0$, $u_y > 0$ then

$$\mathcal{L}(ku) = k\left(\frac{u_x}{\sqrt{1+k^2u_x^2}} + \frac{u_y}{\sqrt{1+k^2u_y^2}}\right) < k\left(\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_y}{\sqrt{1+u_y^2}}\right) = k\mathcal{L}(u).$$

Therefore the PDE is $\boxed{\text{nonlinear and of first order}}$.

(g) If $\mathcal{L}(u) = u_x + e^y u_y$ then the PDE has the form $\mathcal{L}(u) = 0$.

$$\mathcal{L}(u+v) = (u+v)_x + e^y(u+v)_y = u_x + e^y u_y + v_x + e^y v_y = \mathcal{L}(u) + \mathcal{L}(v).$$

$$\mathcal{L}(ku) = (ku)_x + e^y(ku)_y = k\left(u_x + e^y u_y\right) = k\mathcal{L}(u).$$

Therefore the PDE is $\boxed{\text{linear, homogeneous, and of first order}}$.

(h) If $\mathcal{L}(u) = u_t + u_{xxxx} + \sqrt{1+u} = 0$ then the PDE is $\mathcal{L}(u) = 0$.

$$\mathcal{L}(ku) = (ku)_t + (ku)_{xxxx} + \sqrt{1+(ku)} = ku_t + ku_{xxxx} + \sqrt{1+ku}$$

$$k\mathcal{L}(u) = ku_t + ku_{xxxx} + k\sqrt{1+u} = ku_t + ku_{xxxx} + \sqrt{k^2+k^2u} \quad (\text{if } k \geq$$

If $k > 1$ and $u > 0$ then $k^2 + k^2 u > 1 + ku$, so comparing the previous two equations yields $k\mathcal{L}(u) > \mathcal{L}(ku)$.

Therefore the PDE is $\boxed{\text{nonlinear and of fourth order}}$.