Mathematics 325
Homework Assignment 4

Due Date: ____________________

Name: ____________________________

Work exercise 5 on page 24 and exercise 4 on page 27.

5. Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.
   (a) Find the *equilibrium* temperature distribution in the composite rod.
   (b) Sketch it as a function of x in case $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.)

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D.$$

   (a) What can we surely add to any solution to get another solution? So we don't have uniqueness.
   (b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = 0$$

   is a necessary condition for the Neumann problem to have a solution.
   (c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

#5, p. 24

$$\begin{cases} u_t - k_1 u_{xx} \overset{①}{=} 0 & \text{if } 0 < x < L_1,\ t > 0,\ \left(k_1 = \frac{K_1}{c\rho}\right) \\ u_t - k_2 u_{xx} \overset{②}{=} 0 & \text{if } L_1 < x < L_1 + L_2,\ t > 0, \left(k_2 = \frac{K_2}{c\rho}\right) \\ \lim_{x \to L_1^-} u(x,t) \overset{③}{=} \lim_{x \to L_1^+} u(x,t) \text{ and } \lim_{x \to L_1^-} K_1 u_x(x,t) \overset{④}{=} \lim_{x \to L_1^+} K_2 u_x(x,t) & \text{for } t > 0 \\ u(0,t) \overset{⑤}{=} 0 \text{ and } u(L_1 + L_2, t) \overset{⑥}{=} T & \text{for } t > 0. \end{cases}$$

(a) Let $U(x) = \lim_{t \to \infty} u(x,t)$ be the equilibrium solution of the above problem. Then since U is independent of t, from ① and ② we have

$$U(x) = \begin{cases} ax + b & \text{if } 0 < x < L_1, \\ cx + d & \text{if } L_1 < x < L_1 + L_2. \end{cases}$$

14% to here

From ⑤ and ⑥, we have $0 = U(0) = b$ and $T = U(L_1 + L_2) = c(L_1 + L_2) + d$, so

$$U(x) = \begin{cases} ax & \text{if } 0 < x < L_1, \\ c(x - L_1 - L_2) + T & \text{if } L_1 < x < L_1 + L_2. \end{cases}$$

28% to here

Using ③ and ④ yields

$$aL_1 = -cL_2 + T \quad \text{and} \quad K_1 a = K_2 c.$$

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The solution to this system of two linear equations in the two unknowns a and c is:

$$c = \frac{K_1 T}{K_1 L_2 + K_2 L_1}, \quad a = \frac{K_2 T}{K_1 L_2 + K_2 L_1}.$$

56% to here.

That is, the equilibrium temperature distribution in the composite rod is :

$$U(x) = \begin{cases} \dfrac{K_2 T x}{K_1 L_2 + K_2 L_1} & \text{if } 0 \le x \le L_1, \\[2ex] \dfrac{K_1 T (x - L_1 - L_2)}{K_1 L_2 + K_2 L_1} + T & \text{if } L_1 \le x \le L_1 + L_2. \end{cases}$$

(b) Dividing the top and the bottom of the ratios by $c\rho$ in the expression for $U(x)$ in part (a), and using $k_1 = \dfrac{K_1}{c\rho}$ and $k_2 = \dfrac{K_2}{c\rho}$ gives

$$U(x) = \begin{cases} \dfrac{k_2 T x}{k_1 L_2 + k_2 L_1} & \text{if } 0 \le x \le L_1, \\[2ex] \dfrac{k_1 T (x - L_1 - L_2)}{k_1 L_2 + k_2 L_1} + T & \text{if } L_1 \le x \le L_1 + L_2. \end{cases}$$

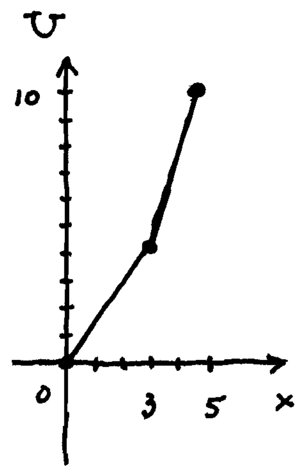
70% to here

Setting $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$ yields

$$U(x) = \begin{cases} \frac{10}{7} x & \text{if } 0 \le x \le 3, \\[2ex] \frac{20}{7}(x-5) + 10 & \text{if } 3 \le x \le 5. \end{cases}$$

84 % to here

The graph of this temperature distribution as a function of x is :



100% to here

#4, p. 27.

$$\begin{cases} \Delta u \overset{①}{=} f(x,y,z) & \text{in } D, \\ \dfrac{\partial u}{\partial n} \overset{②}{=} 0 & \text{on } \partial D. \end{cases}$$

20% (a) Let $u(x,y,z)$ be a solution to ①-②. If C is any constant and $v = u(x,y,z) + C$ then $\Delta(v) = \Delta u$ and $\dfrac{\partial v}{\partial n} = \dfrac{\partial u}{\partial n}$ so v also solves ①-②.

40% (b) Let $u(x,y,z)$ be a solution to ①-②. Then

$$0 = \iint_{\partial D} 0\, dS \overset{\text{by ②}}{=} \iint_{\partial D} \frac{\partial u}{\partial n} dS \overset{\text{definition of } \frac{\partial u}{\partial n}}{=} \iint_{\partial D} \nabla u \cdot \vec{n}\, dS \overset{\text{Divergence Theorem (see p. 393)}}{=} \iiint_D \nabla \cdot (\nabla u)\, dV \overset{\text{definitions}}{=} \iiint_D \Delta u\, dV$$

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$$\overset{\text{by ①}}{=} \iiint_D f(x,y,z)\, dV.$$

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Therefore $\iiint_D f(x,y,z)\,dx\,dy\,dz = 0$ is a necessary condition for ①-② to have a solution.

40% (c) Let us consider ①-② as modeling the steady-state temperature distribution $u(x,y,z)$ at position (x,y,z) in D. The presence of f in ① indicates sources (+) and/or sinks (−) of heat energy within the material occupying D. The condition ② corresponds to an insulated boundary of D. That is, no heat energy flows across ∂D so the system in D is "isolated" or "closed".

The result of part (a) means that, according to the model ①-②, there is no such thing as an "absolute" temperature at position (x,y,z) in D. We are free to arbitrarily assign a particular temperature at a particular point in D, and all temperatures at other points are then measured in reference to this "standard" temperature. (Of course, the model ①-② for steady-state temperatures was derived using the assumptions of continuum mechanics. Quantum mechanical effects leading to an absolute temperature are not incorporated in this model.)

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The result of part (b) means that in order for solutions $u = u(x,y,z)$ to exist for the steady-state temperature distribution problem ①-②, the average value

$$\frac{1}{\mathrm{vol}(D)} \iiint_D f(x,y,z)\,dV$$

20%

of the source/sink term f must be zero, so that the total heat energy of the closed system in D is conserved.