Mathematics 325
Homework Assignment 5

Due Date: ___________________

Name:___________________________

Consider the partial differential equation

(*) $$u_{xx} - 3u_{xt} - 4u_{tt} = 0.$$

(a) Classify the order and type (linear, nonlinear, parabolic, etc.) of (*).
(b) Find the general solution of (*) in the $xt-$ plane, if possible.
(c) Find the solution of (*) that satisfies

$$u(x,0) = x^3 \quad \text{and} \quad u_t(x,0) = -3x^2$$

for $-\infty < x < \infty$.

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$

$\boxed{\text{linear homogeneous}}$

10% (a) $B^2 - 4AC = (-3)^2 - 4(1)(-4) = 25$ $\boxed{\text{hyperbolic}}$ $\boxed{\text{second-order}}$

60% (b) $\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$ 15% to here

Let $\begin{cases} \xi = +4x + t \\ \eta = x - t \end{cases}$ 30% to here. Then $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 4\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$,

i.e. $\frac{\partial}{\partial x} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$. Similarly $\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$. Therefore

$$\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t} = \cancel{4\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 4\left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right) = 5\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = 4\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} + \frac{\partial}{\partial \xi} - \cancel{\frac{\partial}{\partial \eta}} = 5\frac{\partial}{\partial \xi}.$$

45% to here

Thus, the pde is equivalent to $\left(5\frac{\partial}{\partial \eta}\right)\left(5\frac{\partial}{\partial \xi}\right)u = 0 \Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$.
The general solution is $u = f(\xi) + g(\eta)$, i.e. $\boxed{u(x,t) = f(4x+t) + g(x-t)}$
where f and g are arbitrary C^2-functions of a single real variable. 60% to here.

30% (c) $x^3 = u(x,0) = f(4x) + g(x) \Rightarrow 3x^2 = 4f'(4x) + g'(x)$ (1)

$u_t(x,t) = f'(4x+t) - g'(x-t) \Rightarrow -3x^2 = u_t(x,0) = f'(4x) - g'(x)$ (2)

Adding equations (1) and (2) gives $0 = 5f'(4x) \Rightarrow f'(z) = 0 \Rightarrow f(z) = c_1$. 10% to h
Substituting this result in equation (1) gives $3x^2 = g'(x) \Rightarrow g(x) = x^3 + c_2$.
But $x^3 = f(4x) + g(x) = c_1 + x^3 + c_2$ so $c_1 + c_2 = 0$. Thus 20% to here.

$u(x,t) = f(4x+t) + g(x-t) = c_1 + (x-t)^3 + c_2 \Rightarrow \boxed{u(x,t) = (x-t)^3}$.

30% to here.