Mathematics 325 Exam III Name: Dr. Grow

Summer 2001

1. (25 pts.) Use Fourier transform methods to solve

$$u_t - u_{xx} + 3t^2 u = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to $u(x,0) = \phi(x)$ for $-\infty < x < \infty$.

$$\mathcal{F}(u_t - u_{xx} + 3t^2 u)(\xi) = \mathcal{F}(0)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 3t^2\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2 + 3t^2)\mathcal{F}(u)(\xi) = 0$$

← First order linear ode (with parameter ξ) in the variable t. Integrating factor: $e^{\int(\xi^2+3t^2)dt} = e^{\xi^2 t + t^3}$.

$$e^{\xi^2 t + t^3}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+3t^2)e^{\xi^2 t+t^3}\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\left[e^{\xi^2 t+t^3}\mathcal{F}(u)(\xi)\right] = 0$$

$$e^{\xi^2 t+t^3}\mathcal{F}(u)(\xi) = c(\xi)$$

Evaluating at $t=0$ and using the initial condition yields

$$c(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(\varphi)(\xi).$$

Thus $\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t - t^3} = \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t}\cdot e^{-t^3}$

$$= \mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)e^{-t^3} \quad \left(\text{by transform formula I with } a = 1/4t.\right)$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)e^{-t^3}.$$

$$\therefore\ u(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-t^3}\left(\varphi * e^{-\frac{(\cdot)^2}{4t}}\right)(x) = \boxed{\frac{e^{-t^3}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy}.$$

2. (25 pts.) Consider the function $f(x) = x$ on $0 \le x \le 1$.

(a) Calculate the Fourier sine coefficients of f.

(b) Write the Fourier sine series of f.

(c) On the same coordinate axes, sketch the graph of f and the sum of the first three terms of its Fourier sine series.

(d) At which points, if any, of the interval $0 \le x \le 1$ does the Fourier sine series of f converge to f pointwise? Give reasons for your answer.

(e) Use parts (b) and (d) to help show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

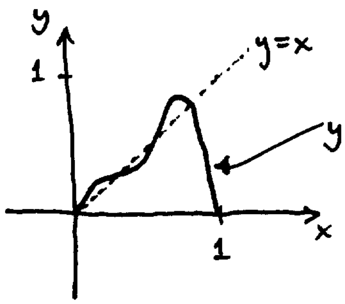
(f) Use part (b) and Parseval's identity to help show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} .$$

(a) $$b_n = \frac{\langle f, \sin(n\pi\cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = 2\int_0^1 x \sin(n\pi x)\,dx = \left.\frac{-2x\cos(n\pi x)}{n\pi}\right|_0^1 + \cancel{2\int_0^1 \frac{\cos(n\pi x)}{n\pi}dx}^{\,0} = \boxed{\frac{2(-1)^{n+1}}{n\pi}}$$

(b) $$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(n\pi x)}{n\pi} \quad \text{on } [0,1] .$$

(c)



$$y = S_3(x) = \frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x)\right)$$

(d) The Fourier sine series of f converges pointwise to $f(x)$ for all $0 \le x < 1$.

Reasons: Since $f(x)=x$ is C^1 on $[0,1]$, Theorem 4(i) implies the F.S.S. of f ~~converges~~ to f pointwise on $0<x<1$. At the endpoints,

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(0)}{n\pi} = 0 = f(0) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(n\pi)}{n\pi} = 0 \ne 1 = f(1) .$$

(e) Take $x = 1/2$ in $x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(n\pi x)}{n\pi}$ for $0 \le x < 1$:

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(n\pi/2)}{n\pi} \iff \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(\pi n/2)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

(f) $$\sum_{n=1}^{\infty} \left|\frac{2(-1)^{n+1}}{n\pi}\right|^2 \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 x^2\,dx \Rightarrow \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} = \frac{1}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3. (25 pts.) Solve $u_{xx} + u_{yy} = 1$ in $1 < x^2 + y^2 < 4$, subject to $u = 0$ on the boundaries $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1$$

If we assume a radial solution [i.e. $u = u(r)$, independent of θ] then $\frac{\partial^2 u}{\partial \theta^2} = 0$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 1 \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r \Rightarrow r\frac{\partial u}{\partial r} = \frac{r^2}{2} + c_1$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{r}{2} + \frac{c_1}{r} \Rightarrow u = \frac{r^2}{4} + c_1 \ln(r) + c_2 .$$

$$0 = u(1) = \frac{1}{4} + c_2 \Rightarrow c_2 = -\frac{1}{4}$$

$$0 = u(2) = 1 + c_1 \ln(2) - \frac{1}{4} \Rightarrow c_1 = \frac{-3}{4\ln(2)}$$

$$\therefore \boxed{u(r,\theta) = \frac{r^2 - 1}{4} - \frac{3\ln(r)}{4\ln(2)}}$$

or

$$\boxed{u(x,y) = \frac{x^2+y^2-1}{4} - \frac{3\ln(x^2+y^2)}{8\ln(2)}}$$

4. (25 pts.) Find a solution to

$$\nabla^2 u = 0$$

in the cube C: $0 < x < \pi$, $0 < y < \pi$, $0 < z < \pi$, given that

$u(x,y,\pi) = [1 + \cos(2x)][1 + \cos(y)]$ for $0 \le x \le \pi$, $0 \le y \le \pi$,

and that u satisfies homogeneous Neumann boundary conditions on the other five faces of C.

BONUS (10 pts): Can there be more than one solution to the above problem? Give reasons for your answer.

$u(x,y,z) = X(x)Y(y)Z(z)$ in $u_{xx}+u_{yy}+u_{zz}=0$ gives $X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \mu \quad \Rightarrow \quad \frac{Z''(z)}{Z(z)} - \mu = -\frac{Y''(y)}{Y(y)} = \nu .$$

$$\therefore \begin{cases} X''(x) + \mu X(x) = 0, & X'(0) = 0 = X'(\pi) \\ Y''(y) + \nu Y(y) = 0, & Y'(0) = 0 = Y'(\pi) \\ Z''(z) - (\mu+\nu) Z(z) = 0, & Z'(0) = 0 \end{cases} \Rightarrow \begin{cases} \mu_m = m^2, \ X_m(x) = \cos(mx) & (m=0,1,2,\dots) \\ \nu_n = n^2, \ Y_n(y) = \cos(ny) & (n=0,1,2,\dots) \\ Z_{m,n}(z) = \cosh(z\sqrt{m^2+n^2}) \end{cases}$$

$$u(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}\cos(mx)\cos(ny)\cosh(z\sqrt{m^2+n^2})$$

$$\underbrace{[1+\cos(2x)][1+\cos(y)]}_{1+\cos(2x)+\cos(2x)\cos(y)+\cos(y)} = u(x,y,\pi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{m,n}\cos(mx)\cos(ny) \qquad \left(B_{m,n} = A_{m,n}\cosh(\pi\sqrt{m^2+n^2}\right.$$

By inspection, we should have $1 = B_{0,0}$, $1 = B_{0,1}$, $1 = B_{2,0}$, $1 = B_{2,1}$ and all other $B_{m,n} = 0$. I.e. $A_{0,0} = 1$, $A_{0,1} = \frac{1}{\cosh(\pi)}$, $A_{2,0} = \frac{1}{\cosh(2\pi)}$, $A_{2,1} = \frac{1}{\cosh(\pi\sqrt{5})}$, and all other $A_{m,n} = 0$.

$$\therefore \boxed{u(x,y,z) = 1 + \frac{\cos(y)\cosh(z)}{\cosh(\pi)} + \frac{\cos(2x)\cosh(2z)}{\cosh(2\pi)} + \frac{\cos(2x)\cos(y)\cosh(z\sqrt{5})}{\cosh(\pi\sqrt{5})}}$$