Mathematics 325 Exam III Spring 2002 Name: Dr. Grow (2 pts.)

3.(26 pts.) Consider the 2π-periodic function f given on one period by $f(x) = x^2$ if $-\pi \le x < \pi$.

(a) Calculate the full Fourier series of f on $[-\pi,\pi]$.

(b) Write the sum of the first three nonzero terms of the full Fourier series of f and sketch the graph of this sum on $[-\pi,\pi]$. On the same coordinate axes, sketch the graph of f.

(c) Does the full Fourier series of f converge to f in the mean square sense on $[-\pi,\pi]$? Why?

(d) Does the full Fourier series of f converge to f pointwise on $[-\pi,\pi]$? Why?

(e) Does the full Fourier series of f converge to f uniformly on $[-\pi,\pi]$? Why?

(f) Use the results above to help find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

(g) Use the results above to help find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

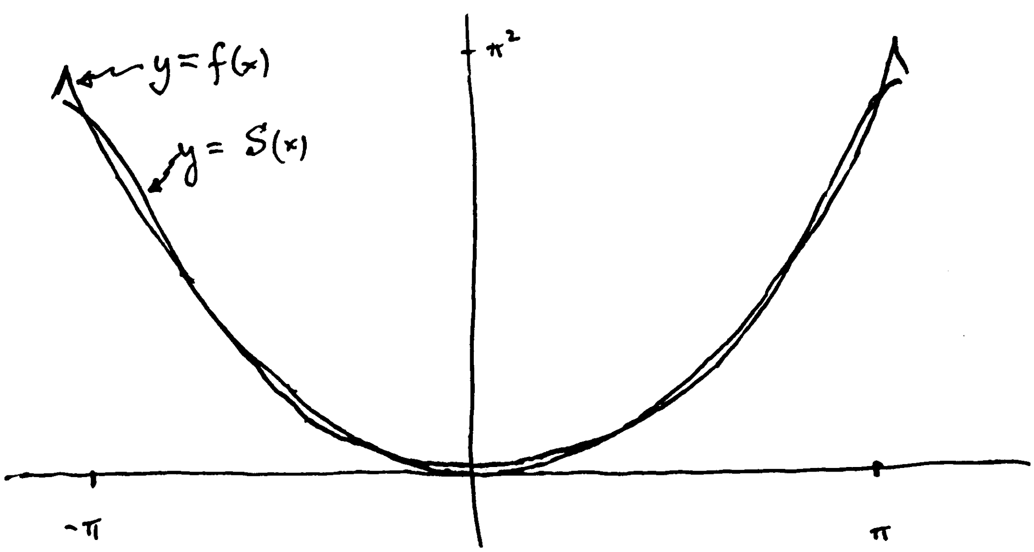
(a) Since f is even, the full Fourier series of f is a cosine series, i.e. $b_n = 0$ for all $n \ge 1$.

For $n \ge 1$, $$a_n = \frac{\langle f, \cos(n\cdot)\rangle}{\langle \cos(n\cdot), \cos(n\cdot)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{u} \underbrace{\cos(nx)dx}_{dv} = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx = \cancel{\frac{2}{\pi}x^2\frac{\sin(nx)}{n}\Big|_0^{\pi}} - \frac{2}{\pi n}\int_0^{\pi} \underbrace{2x}_{u}\,\underbrace{\sin(nx)dx}_{dv}$$

$$= \frac{4x}{\pi n^2}\cos(nx)\Big|_0^{\pi} - \cancel{\frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx} = \frac{4(-1)^n}{n^2}. \quad a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3}.$$

$$\therefore \boxed{f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2}}.$$

(b) $S(x) = \frac{\pi^2}{3} - 4\cos(x) + \cos(2x)$



(c) Since $\int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi}^{\pi} (x^2)^2\,dx = \frac{2\pi^5}{5} < \infty$, the L^2 convergence theorem (3) implies the full Fourier series of f converges to f in the mean square sense on $(-\pi,\pi)$.

(d) Since f is continuous and f' is piecewise continuous and both are 2π-periodic, Theorem 4^∞ implies the full Fourier series of f converges pointwise to $f(x)$ for all $x \in (-\infty,\infty)$.

(e) Yes, the full Fourier series of f converges uniformly to f on $[-\pi,\pi]$, although the uniform convergence theorem (2) doesn't apply. (The function f does not satisfy the second periodic boundary condition $\varphi'(-\pi) = \varphi'(\pi)$.) To see this,

$$\max_{-\pi\le x\le\pi} \left| f(x) - \frac{\pi^2}{3} - \sum_{n=1}^{N} \frac{4(-1)^n\cos(nx)}{n^2} \right| \overset{\text{by part (d)}}{=} \max_{-\pi\le x\le\pi} \left| \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2} - \frac{\pi^2}{3} - \sum_{n=1}^{N} \frac{4(-1)^n\cos(nx)}{n^2} \right|$$

$$= \max_{-\pi\le x\le\pi} \left| \sum_{n=N+1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2} \right| \le \sum_{n=N+1}^{\infty} \frac{4}{n^2} \longrightarrow 0 \text{ as } N\to\infty$$

(since it is the "tail" of the series $4\sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges by the p-series test with $p=2$).

(f) $$0 = f(0) \overset{\text{by part (d)}}{=} \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \overbrace{\cos(n0)}^{1}}{n^2}, \text{ so } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \left(\frac{-\pi^2}{3}\right)\frac{1}{4} = \boxed{\frac{-\pi^2}{12}}$$

(g) By Parseval's identity $\sum_{m=0}^{\infty} |A_m|^2 \int_a^b |X_m(x)|^2\,dx = \int_a^b |\varphi(x)|^2\,dx$, we have

$$\int_{-\pi}^{\pi} 1^2\,dx \cdot \left|\frac{\pi^2}{3}\right|^2 + \sum_{n=1}^{\infty} \left|\frac{4(-1)^n}{n^2}\right|^2 \int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} |f(x)|^2\,dx$$

$$\Rightarrow \frac{2\pi^5}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}\cdot\pi = \frac{2\pi^5}{5}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{2\pi^5}{5} - \frac{2\pi^5}{9}\right)\frac{1}{16\pi} = \frac{8\pi^5}{45}\cdot\frac{1}{16\pi} = \boxed{\frac{\pi^4}{90}}$$

4. (26 pts.) Let u be the solution to the problem

$$\nabla^2 u = 0 \text{ in the disk } D = \{ (r;\theta) : 0 \le r < 2,\ -\pi \le \theta < \pi \},$$
$$u(2;\theta) = 3\sin(2\theta) + 1 \text{ for } -\pi \le \theta < \pi.$$

(a) Find the maximum value of u in

$$\bar{D} = \{ (r;\theta) : 0 \le r \le 2,\ -\pi \le \theta < \pi \}.$$

(b) Calculate the value of u at the origin.

(Hint: These problems can be answered without computing an explicit formula for u as a function of r and θ.)

(a) By the maximum principle, the maximum value of u occurs on the circumference of the disk: $\partial D = \{ (2;\theta) : -\pi \le \theta < \pi \}$. For all $\theta \in [-\pi, \pi)$ we have

$$u(2;\theta) = 3\sin(2\theta)+1 \le 3\sin\left(2\left(\tfrac{\pi}{4}\right)\right)+1 = 3\sin\left(\tfrac{\pi}{2}\right)+1 = 3\cdot 1+1 = \boxed{4}.$$

(b) By the mean value theorem, the value of u at the origin is equal to the average value of u on the circumference of the disk, ∂D. Thus

$$u(0;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(2;\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(3\sin(2\theta)+1\right)d\theta = \frac{1}{2\pi}\left(-\frac{3\cos(2\theta)}{2}+\theta\right)\Bigg|_{-\pi}^{\pi} = \boxed{1}.$$

5. (26 pts.) Find the steady-state temperature distribution inside an annular plate with inner radius 1 and outer radius 2 if the outer edge $r = 2$ is insulated and on the inner edge $r = 1$ the temperature is maintained as θ^2 for $-\pi \le \theta < \pi$. (Hint: You should find the results of problem 3 useful.)

We need to solve
$$\begin{cases} ① \ \nabla^2 u = 0 \quad \text{in } A = \{(r;\theta): 1 < r < 2, -\pi \le \theta < \pi\}, \\ ② \ u_r(2;\theta) = 0 \quad \text{for } -\pi \le \theta < \pi, \\ ③ \ u(1;\theta) = \theta^2 \quad \text{for } -\pi \le \theta < \pi; \end{cases}$$

we also have the implied boundary conditions $u(r;-\pi) \overset{④}{=} u(r;\pi)$ and $u_\theta(r;-\pi) \overset{⑤}{=} u_\theta(r;\pi)$ for $1 \le r \le 2$.

We seek nontrivial solutions to the homogeneous part of the problem ①-②-④-⑤ of the form $u(r;\theta) = R(r)\Theta(\theta)$. Substituting in ①-②-④-⑤ and simplifying yields

$$0 = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)$$

$$\Rightarrow \quad \frac{r^2 R''(r) + rR'(r)}{R(r)} = \frac{-\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$$0 = u_r(2;\theta) = R'(2)\Theta(\theta), \qquad 0 = u(r;-\pi) - u(r;\pi) = R(r)[\Theta(-\pi) - \Theta(\pi)]$$

$$0 = u_\theta(r;-\pi) - u_\theta(r;\pi) = R(r)[\Theta'(-\pi) - \Theta'(\pi)].$$

Thus
$$\begin{cases} \boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi)} \leftarrow \text{Eigenvalue Problem} \\ r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \quad R'(2) = 0 \end{cases}$$

The eigenvalues/eigenfunctions are

$$\lambda_0 = 0, \quad \Theta_0(\theta) = a_0,$$

$$\lambda_n = n^2, \quad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta) \qquad (n = 1,2,3,\ldots)$$

and a solution to the radial problem $r^2 R_n''(r) + rR_n'(r) - n^2 R_n(r) = 0$ is of the form $R_n(r) = r^\alpha$ where α is a constant. Then $R_n'(r) = \alpha r^{\alpha-1}$, $R_n''(r) = \alpha(\alpha-1)r^{\alpha-2}$

so $\quad r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = 0 \Rightarrow \alpha(\alpha-1) + \alpha - n^2 = 0 \Rightarrow \alpha = \pm n.$

If $n \ge 1$, then $R_n(r) = cr^n + dr^{-n}$ and $0 = R_n'(2)$ imply $0 = nc2^{n-1} - nd2^{-n-1}$, so $d = c2^{2n}$; i.e. $R_n(r) = r^n + 2^{2n} r^{-n}$ (up to a constant factor).

If $n=0$ then the general solution to $r^2 R_0''(r) + r R_0'(r) = 0$ is found as follows: $(r R_0'(r))' = r R_0''(r) + R_0'(r) = 0 \Rightarrow r R_0'(r) = c \Rightarrow R_0(r) = \int \frac{c}{r} dr$ $= c \ln(r) + d$. Then $0 = R_0'(2) = \frac{c}{2} \Rightarrow R_0(r) = 1$ (up to a constant factor).

A formal solution to ①-②-④-⑤ is $u(r;\theta) = \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) =$ $a_0 + \sum_{n=1}^{\infty} (r^n + 2^{2n} r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta))$. We want to choose the arbitrary coefficients so ③ is satisfied:

$$\theta^2 = u(1;\theta) = a_0 + \sum_{n=1}^{\infty} (2^{2n}+1)(a_n \cos(n\theta) + b_n \sin(n\theta)) \quad \text{for } -\pi \le \theta < \pi .$$

By problem #3(d),

$$\theta^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\theta)}{n^2} \quad \text{for all } -\pi \le \theta < \pi .$$

Therefore $a_0 = \frac{\pi^2}{3}$, $\frac{4(-1)^n}{n^2} = (2^{2n}+1)a_n$, and $b_n = 0$ for $n \ge 1$.

Consequently,

$$\boxed{u(r;\theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\theta)\left(r^n + 2^{2n} r^{-n}\right)}{n^2\left(1 + 2^{2n}\right)}} .$$

Mathematics 325
Winter 2002
Two Take-Home Problems for Exam III

1. (10 pts.) (a) Let n be a nonnegative integer. Show that the operator T given by

$$Tf(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2} f(r) \qquad (0 < r \le 1)$$

is hermitian on the vector space

$$V_B = \{ f \in C^2(0,1] : f(1) = 0, \ f \text{ and } f' \text{ bounded on } (0,1] \}$$

equipped with the inner product

$$(*) \qquad \langle f,g\rangle = \int_0^1 f(r)\overline{g(r)}\,r\,dr.$$

(b) Are the eigenvalues of T on V_B real numbers?

(c) Are the eigenvalues of T on V_B positive?

(d) Are the eigenfunctions of T on V_B, corresponding to distinct eigenvalues, orthogonal on (0,1) relative to the inner product (*)?

(Please give reasons for your answers to (b)-(d).)

. (10 pts.) Use separation of variables to solve the variable density vibrating string problem:

$$\frac{1}{(1+x)^2} u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0 = u(1,t) \qquad \text{for } 0 \le t < \infty,$$

$$u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \qquad \text{for } 0 \le x \le 1.$$