Mathematics 325 Exam I Name: Dr. Grow

Fall 2000

(20 pts.) (a) Find the general solution of

(*) $(1 + x^2)u_x + xyu_y = 0$

in the xy-plane.

(b) Sketch and identify some characteristic curves of (*).

(a) $D_{[1+x^2,\, xy]}u = [1+x^2, xy]\cdot\nabla u = (1+x^2)u_x + xyu_y = 0$. Therefore $u = u(x,y)$ is constant along curves whose tangent line at a general point (x,y) is parallel to $[1+x^2, xy]$; i.e. along curves satisfying

$$\frac{dy}{dx} = \frac{xy}{1+x^2}.$$

Thus, $\dfrac{dy}{y} = \dfrac{x\,dx}{1+x^2}$, so $\ln|y| = \int \dfrac{dy}{y} = \dfrac{1}{2}\int \dfrac{2x\,dx}{1+x^2} = \dfrac{1}{2}\ln|1+x^2| + c_1$.

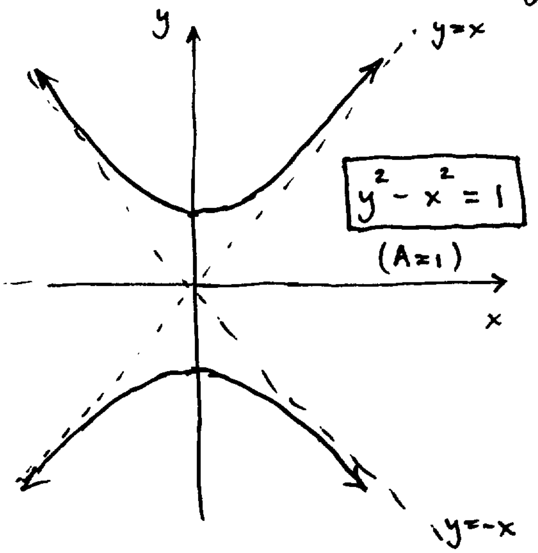
That is, $\ln y^2 = \ln(1+x^2) + c_2$, so $y^2 = A(1+x^2)$ $\quad (A = e^{c_2})$.

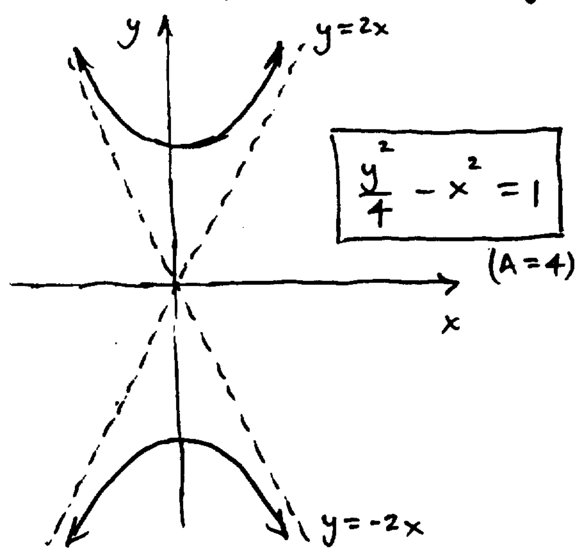
At a general point (x,y) on such a curve,

$$u(x,y) = u(x, \pm\sqrt{A(1+x^2)}) \overset{\text{set } x=0}{=} u(0, \pm\sqrt{A}) = f(A).$$

Therefore, the general solution of (*) is $\boxed{u(x,y) = f\left(\frac{y^2}{1+x^2}\right)}$ where f is a C^1-function of a single real variable.

(b) The characteristic curves of (*) are $\boxed{y^2 = A(1+x^2)}$, and these are hyperbolas.





2. (20 pts.) (a) Find the general solution of

(**) $u_x + 2u_y + 7(2x - y)u = 7(2x - y)(x + 2y)$

in the xy-plane.

(b) Sketch and identify some characteristic curves of (**).

Let $\begin{cases} \xi = 2x - y, \\ \eta = x + 2y. \end{cases}$ Then $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = 2\dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta}$

and $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y} = -\dfrac{\partial u}{\partial \xi} + 2\dfrac{\partial u}{\partial \eta}.$

Substituting these expressions into (*) yields

$$\left(\cancel{2\frac{\partial u}{\partial \xi}} + \frac{\partial u}{\partial \eta}\right) + 2\left(\cancel{-\frac{\partial u}{\partial \xi}} + 2\frac{\partial u}{\partial \eta}\right) + 7\xi u = 7\xi\eta$$

$$\frac{\partial u}{\partial \eta} + \frac{7}{5}\xi u = \frac{7}{5}\xi\eta \longleftarrow$$

1st-order, linear in the variable η.
Integrating factor: $e^{\int \frac{7}{5}\xi\, d\eta} = e^{\frac{7}{5}\xi\eta}$

$$e^{\frac{7}{5}\xi\eta}\frac{\partial u}{\partial \eta} + \frac{7}{5}\xi e^{\frac{7}{5}\xi\eta} u = \frac{7}{5}\xi\eta e^{\frac{7}{5}\xi\eta}$$

$$\frac{\partial}{\partial \eta}\left(e^{\frac{7}{5}\xi\eta} u\right) = \frac{7}{5}\xi\eta e^{\frac{7}{5}\xi\eta}$$

$$e^{\frac{7}{5}\xi\eta} u = \int \underbrace{\frac{7}{5}\xi\eta}_{v}\, \underbrace{e^{\frac{7}{5}\xi\eta}\, d\eta}_{dV} \qquad \text{(Parts)}$$

$$= \frac{7}{5}\xi\eta\left(\frac{5}{7\xi}\right)e^{\frac{7}{5}\xi\eta} - \int \frac{5}{7\xi}e^{\frac{7}{5}\xi\eta}\,\frac{7}{5}\xi\, d\eta$$

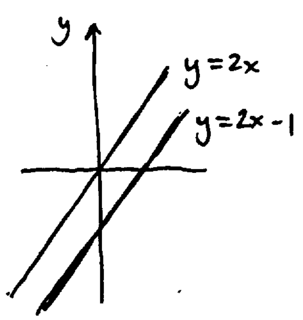
$$= \eta e^{\frac{7}{5}\xi\eta} - \frac{5}{7\xi}e^{\frac{7}{5}\xi\eta} + c(\xi)$$

$$\therefore\ u = \eta - \frac{5}{7\xi} + c(\xi)e^{-\frac{7}{5}\xi\eta}$$

(a) $$\boxed{u(x,y) = x + 2y - \frac{5}{7(2x-y)} + f(2x-y)e^{-\frac{7}{5}(2x-y)(x+2y)}}$$ where f is a C^1-function of a single real variable.

(b) $\dfrac{dy}{dx} = 2 \Rightarrow \boxed{y = 2x + c}$

Characteristic curves of (**).



For your information, here is the general solution to the original #2(a).

$$(**) \qquad u_x + 2u_y + 7(2x-y)u = 7(2x-y)(x+3y)$$

Let $\begin{cases} \xi = x+3y \\ \eta = 2x-y. \end{cases}$

Then $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = 3\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$, so $\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} = 7\frac{\partial}{\partial \xi}$.

Thus $(**)$ is equivalent to

$$7\frac{\partial u}{\partial \xi} + 7\eta u = 7\eta\xi \qquad \text{or} \qquad \frac{\partial u}{\partial \xi} + \eta u = \eta\xi .$$

An integrating factor for this first-order linear equation is $e^{\int \eta d\xi} = e^{\eta\xi}$.

$$e^{\eta\xi}\frac{\partial u}{\partial \xi} + \eta e^{\eta\xi} u = \eta\xi e^{\eta\xi}$$

$$\frac{\partial}{\partial \xi}\left(e^{\eta\xi} u\right) = \eta\xi e^{\eta\xi}$$

$$\therefore \quad e^{\eta\xi} u = \int \underbrace{\eta\xi}_{v} \underbrace{e^{\eta\xi} d\xi}_{dV} = \eta\xi \frac{e^{\eta\xi}}{\eta} - \int \frac{e^{\eta\xi}}{\eta}\eta \, d\xi = \left(\xi - \frac{1}{\eta}\right)e^{\eta\xi} + c(\eta)$$

$$\therefore \ u = \xi - \frac{1}{\eta} + c(\eta)e^{-\eta\xi}$$

$$\boxed{u(x,y) = x+3y - \frac{1}{2x-y} + f(2x-y)e^{-(2x-y)(x+3y)}}$$

where f is a C'-function of a single real variable.

3. (20 pts.) (a) Find the most general quadratic polynomial function in the two independent variables x and t, i.e.

$$u(x,t) = Ax^2 + Bxt + Ct^2 + Dx + Et + F \qquad (A, \ldots, F \text{ constants}),$$

which is a solution to

$$(***) \qquad u_t - 2u_{xx} = 0 \qquad \text{for } -\infty < x < \infty \text{ and } -\infty < t < \infty.$$

(b) Use part (a) to help find a solution to (***) which satisfies the initial condition $u(x,0) = x^2$ for $-\infty < x < \infty$.

$$u_t = Bx + 2Ct + E$$
$$u_x = 2Ax + Bt + D$$
$$u_{xx} = 2A$$

Substituting in (***) gives

$$Bx + 2Ct + E - 4A = 0 \quad \text{for all } x \text{ and } t.$$

Equating coefficients of the polynomial on the left side of this equation to zero gives

$$B = 0 = 2C = E - 4A.$$

Therefore, the most general quadratic polynomial function in x and t that solves (***) is

$$\boxed{u(x,t) = Ax^2 + Dx + 4At + F}$$ (where A, D, and F are arbitrary constants).

(b) We assume that the solution to the I.V.P. is a quadratic polynomial function in x and t. By part (a),

$$x^2 \overset{\text{want}}{=} u(x,0) = Ax^2 + Dx + F \quad \text{for all real } x.$$

Equating coefficients of the polynomials on the left and right sides of this equation yields

$$1 = A \text{ and } D = F = 0.$$

Thus, using part (a),

$$\boxed{u(x,t) = x^2 + 4t}$$

is a solution of the I.V.P.

4. (20 pts.) Let $u = u(x,y,z,t)$ denote the temperature at time $t \geq 0$ at each point (x,y,z) of a homogeneous body occupying the spherical region

$$B = \{ (x,y,z) : x^2 + y^2 + z^2 \leq 25 \}.$$

The body is completely insulated, and the initial temperature at each point is equal to its distance from the center of B.

(a) Write (without proof) the partial differential equation and the complete initial/boundary conditions that govern the temperature function.

(b) Use Gauss' divergence theorem to help show that the heat energy

$$H(t) = \iiint_B c\rho u(x,y,z,t)\,dx\,dy\,dz$$

of the body at time t is actually a constant function of time. (Here c and ρ denote the (constant) specific heat and density, respectively, of the material in B.)

(c) Compute the (constant) steady-state temperature that the body reaches after a long time.

(a) ① $u_t - k(\overbrace{u_{xx} + u_{yy} + u_{zz}}^{\nabla^2 u}) = 0$ for all $x^2+y^2+z^2 < 25$ and all $t > 0$,

② $\nabla u \cdot \vec{n} = 0$ for all $x^2+y^2+z^2 = 25$ and all $t \geq 0$,

(Here $\vec{n} = (x\hat{i} + y\hat{j} + z\hat{k})/5$ is the outward unit normal to the boundary of B.)

③ $u(x,y,z,0) = \sqrt{x^2+y^2+z^2}$ for all $x^2+y^2+z^2 \leq 25$.

(b) $$\frac{dH}{dt} = \iiint_B c\rho u_t(x,y,z,t)\,dx\,dy\,dz \overset{\text{by ①}}{=} \iiint_B c\rho k \nabla^2 u(x,y,z,t)\,dx\,dy\,dz$$

$$\overset{\text{by Gauss' Div. Thm.}}{=} \iint_{\partial B} c\rho k \nabla u(x,y,z,t)\cdot \vec{n}(x,y,z)\,dS \overset{\text{by ②}}{=} 0 \text{ for all } t > 0.$$

Thus, $H = H(t)$ is a constant function for $t \geq 0$.

(c) Let U denote the steady-state temperature of the body; i.e. $U = \lim_{t\to\infty} u(x,y,z,t)$ for all $x^2+y^2+z^2 < 25$. Then

$$\iiint_B u(x,y,z,t)\,dx\,dy\,dz = \frac{H(t)}{c\rho} = \text{constant} = \frac{H(0)}{c\rho} = \iiint_B u(x,y,z,0)\,dx\,dy\,dz.$$

Taking the limit as $t \to \infty$ in the above identity gives

$$U \,\text{vol}(B) = \iiint_B U\,dx\,dy\,dz = \iiint_B \lim_{t\to\infty} u(x,y,z,t)\,dx\,dy\,dz = \lim_{t\to\infty} \iiint_B u(x,y,z,t)\,dx\,dy\,dz$$

$$= \iiint_B u(x,y,z,0)\,dx\,dy\,dz,$$

(OVER)

#4 (c) (cont.)

$$\therefore \quad v = \frac{\iiint_B u(x,y,z,0)\,dx\,dy\,dz}{vol(B)} = \frac{\iiint_B \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz}{vol(B)}$$

$$= \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{5} r \cdot r^2 \sin\varphi \, dr\, d\varphi\, d\theta}{\frac{4}{3}\pi(5)^3}$$

$$= \frac{\left(\frac{1}{4}r^4 \Big|_0^5\right)\left(-\cos\varphi \Big|_0^{\pi}\right)\left(\theta \Big|_0^{2\pi}\right)}{\frac{4}{3}\pi(5)^3}$$

$$= \frac{\frac{1}{4}(5)^4 \cdot 2 \cdot 2\pi}{\frac{4}{3}\pi(5)^3}$$

$$= \boxed{\frac{15}{4}}$$

5. (20 pts.) (a) Classify the partial differential equation

(+) $$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$

as elliptic, parabolic, or hyperbolic.

(b) Derive the general solution to (+) in the xt-plane.

(c) Find the solution to (+) in the xt-plane which satisfies the initial conditions

$$u(x,0) = e^x \quad \text{and} \quad u_t(x,0) = 40x - e^x \quad \text{for } -\infty < x < \infty.$$

(a) $B^2 - 4AC = (-3)^2 - 4(1)(-4) = 25 > 0$ $\boxed{\text{hyperbolic}}$

(b) $\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$ Let $\begin{cases} \xi = 4x + t \\ \eta = x - t \end{cases}$

Then $\frac{\partial}{\partial x} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ (by the chain rule) so

$\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 4\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) = 5\frac{\partial}{\partial \eta}$ and

$\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} = 5\frac{\partial}{\partial \xi}$. Thus, the p.d.e. is equivalent to

$$\left(5\frac{\partial}{\partial \eta}\right)\left(5\frac{\partial}{\partial \xi}\right)u = 0, \quad \text{i.e.} \quad \frac{\partial^2 u}{\partial \eta \partial \xi} = 0.$$

Thus $\frac{\partial u}{\partial \xi} = c_1(\xi)$ and $u = \int c_1(\xi)d\xi + c_2(\eta)$; i.e.

$$u = f(\xi) + g(\eta) \quad \text{or} \quad \boxed{u(x,t) = f(4x+t) + g(x-t)}$$

where f and g are C^2-functions of a single real variable.

(c) ① $e^x = u(x,0) = f(4x) + g(x)$ $(-\infty < x < \infty)$

$u_t(x,t) = f'(4x+t) - g'(x-t)$

② $40x - e^x = u_t(x,0) = f'(4x) - g'(x)$ $(-\infty < x < \infty)$

Differentiating ① gives

③ $e^x = 4f'(4x) + g'(x)$ $(-\infty < x < \infty)$

(OVER)

#5 (c) (cont.) Adding ② and ③ yields

④ $$40x = 5f'(4x) \quad (-\infty < x < \infty),$$

and consequently, integrating ④ over the interval $0 \le \xi \le x$,

Let $w = 4\xi$, then $dw = 4d\xi$

$$4x^2 = \int_0^x 8\xi\, d\xi = \int_0^x f'(4\xi)\, d\xi = \frac{1}{4}\int_0^{4x} f'(w)\,dw = \frac{1}{4}\left[f(4x) - f(0)\right]$$

$$16x^2 = f(4x) - f(0)$$

$$v^2 = f(v) - f(0) \qquad (-\infty < v < \infty)$$

Substituting in ① gives

$$g(x) = e^x - f(4x) = e^x - 16x^2 - f(0) \qquad (-\infty < x < \infty),$$

Thus

$$u(x,t) = f(4x+t) + g(x-t)$$

$$u(x,t) = (4x+t)^2 + \cancel{f(0)} + e^{x-t} - 16(x-t)^2 - \cancel{f(0)}$$

$$\boxed{u(x,t) = (4x+t)^2 + e^{x-t} - 16(x-t)^2}$$

(A little algebra shows

$$u(x,t) = 40xt - 15t^2 + e^{x-t}.\ )$$