Mathematics 325 Exam I Summer 2006 Name: Dr. Grow

1.(20 pts.) Solve $3t^2u_x + 2xu_t = 0$ subject to the initial condition $u(x,0) = x^6$ for all real x. Sketch the region in the xt-plane where the solution to this initial value problem is uniquely determined.

$$\langle 3t^2, 2x\rangle \cdot \langle u_x, u_t\rangle = 0$$

$$D_{\langle 3t^2, 2x\rangle} u = 0$$

By the geometric method, the solution is constant along curves whose tangent line at (x,t) is parallel to $\langle 3t^2, 2x\rangle$. That is, along curves satisfying

5 $$\frac{dt}{dx} = \frac{b(x,t)}{a(x,t)} = \frac{2x}{3t^2}.$$ (Characteristic curves of the PDE

9 $$\therefore \quad 3t^2 dt = 2x\,dx \Rightarrow t^3 = x^2 + c \Rightarrow t^3 - x^2 = c.$$

Along such a curve u is constant:

$$u(x,t) = u\left(x, (x^2+c)^{1/3}\right) = u\left(0, (0+c)^{1/3}\right) = f(c).$$

Therefore the general solution of the PDE is

13 $u(x,t) = f(t^3 - x^2)$ where f is a C^1-function of a single real variable

We want to choose f so that

$$x^6 = u(x,0) = f(0^3 - x^2) = f(-x^2) \quad \text{for all } -\infty < x < \infty.$$

$$\therefore \quad -(-x^2)^3 = f(-x^2) \text{ for all real } x \iff -z^3 = f(z) \text{ for all } z \le 0.$$

18 Thus, $\boxed{u(x,t) = -(t^3 - x^2)^3}$ is the particular solution of the IVP.

This solution is uniquely determined in the region where $t^3 - x^2 \le 0 \iff t \le x^{2/3}$

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2.(20 pts.) Find the general solution of $u_x + 2u_y + 5u = 2x - y$ in the xy-plane.

We use the change-of-coordinates method. Let $\begin{cases} \xi = x + 2y \\ \eta = 2x - y \end{cases}$.

$$\left.\begin{aligned} \text{Then } \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial x}}^{1} + \frac{\partial u}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial x}}^{2} = u_\xi + 2u_\eta \\ \text{and } \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial y}}^{2} + \frac{\partial u}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial y}}^{-1} = 2u_\xi - u_\eta \end{aligned}\right\} \Rightarrow u_x + 2u_y = 5u_\xi.$$

Therefore

8 the PDE is equivalent to $5u_\xi + 5u = \eta \Rightarrow u_\xi + u = \frac{\eta}{5}$. This is a first-order linear ODE in ξ (with parameter η). An integrating factor is $e^{\int p\, d\xi} = e^{\int 1\, d\xi} = e^{\xi}$.

16 $e^{\xi}u_\xi + e^{\xi}u = \frac{\eta}{5}e^{\xi} \Rightarrow \frac{\partial}{\partial \xi}(e^{\xi}u) = \frac{\eta}{5}e^{\xi} \Rightarrow e^{\xi}u = \int \frac{\eta}{5}e^{\xi}d\xi = \frac{\eta}{5}e^{\xi} + c(\eta)$.

$\therefore\ u = \frac{\eta}{5} + c(\eta)e^{-\xi}$. The general solution in the xy-plane is

$$\boxed{u(x,y) = \frac{2x-y}{5} + f(2x-y)e^{-(x+2y)}}$$

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where f is an arbitrary C^1-function of a single real variable.

3.(20 pts.) Classify the type (hyperbolic, parabolic, elliptic) of the linear second order partial differential equation $u_{xx} - 2u_{xy} + u_{yy} - u = 0$ and find, if possible, its general solution in the xy-plane.

4 $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$ The equation is of $\boxed{\text{parabolic}}$ type.

The PDE can be expressed in operator form as

6 $$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u - u = 0.$$

10 Let $\begin{cases} \xi = +x + y \\ \eta = x - y. \end{cases}$ Then $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial x}^{1} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial x}^{1} = \dfrac{\partial v}{\partial \xi} + \dfrac{\partial v}{\partial \eta} \Rightarrow \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$

and $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial y}^{1} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial y}^{-1} = \dfrac{\partial v}{\partial \xi} - \dfrac{\partial v}{\partial \eta} \Rightarrow \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial \xi} - \dfrac{\partial}{\partial \eta}$

Therefore $\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial \eta}$ so the PDE is equivalent to $\left(2\dfrac{\partial}{\partial \eta}\right)^2 u - u = 0$

14 $\Rightarrow 4\dfrac{\partial^2 u}{\partial \eta^2} - u = 0 \Rightarrow \dfrac{\partial^2 u}{\partial \eta^2} - \dfrac{1}{4}u = 0$. This is a second-order ODE in η (with implicit parameter ξ) with constant coefficients. $u = e^{r\eta}$ leads to $r^2 - \frac{1}{4} = 0 \Rightarrow$

18 $r = \pm\frac{1}{2}$. Therefore $u = c_1(\xi)e^{\frac{1}{2}\eta} + c_2(\xi)e^{-\frac{1}{2}\eta}$ so the general solution of the PDE in the xy-plane is $\boxed{u(x,y) = f(x+y)e^{\frac{1}{2}(x-y)} + g(x+y)e^{-\frac{1}{2}(x-y)}}$ where f and

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g are arbitrary C^2-functions of a single real variable.

4.(20 pts.) **Write the solution of the initial value problem**

$$u_{tt} - 4u_{xx} = 0 \quad \text{for} \quad -\infty < x < \infty,\ 0 < t < \infty,$$

$$u(x,0) = e^x \quad \text{and} \quad u_t(x,0) = 4 + x \quad \text{for} \quad -\infty < x < \infty.$$

We use d'Alembert's formula

6 $$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$$

with $c = 2$, $\varphi(x) = e^x$, and $\psi(x) = 4 + x$. Therefore

10 $$u(x,t) = \frac{1}{2}\left[e^{x+2t} + e^{x-2t}\right] + \frac{1}{4}\int_{x-2t}^{x+2t}(4+\xi)\,d\xi$$

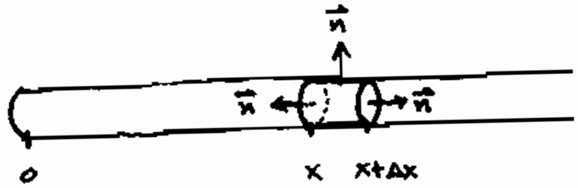
$$= e^x\left(\frac{e^{2t} + e^{-2t}}{2}\right) + \left(\xi + \frac{1}{8}\xi^2\right)\Big|_{x-2t}^{x+2t}$$

18 $$= e^x\cosh(2t) + \cancel{x}+2t + \frac{1}{8}(x+2t)^2 - (\cancel{x}-2t) - \frac{1}{8}(x-2t)^2$$

$$= e^x\cosh(2t) + 4t + \frac{1}{8}\Big[\overbrace{(x+\cancel{2t}+x-\cancel{2t})(\cancel{x}+2t-(\cancel{x}-2t))}^{2x\cdot 4t}\Big]$$

20 $$\boxed{u(x,t) = e^x\cosh(2t) + 4t + xt}$$

5.(20 pts.) Consider heat flow in a long thin rod suspended horizontally in a large room with constant ambient temperature T_0. On the sides of the rod, convective heat exchange takes place according to Newton's law of cooling - the velocity of the flow is proportional to the temperature difference. Heat is conducted down the axis of the rod according to Fourier's law - the velocity of the flow is proportional to the temperature gradient. Derive the equation satisfied by the temperature $u(x,t)$ at position x units from the left end of the rod and at time t, neglecting the temperature variation on cross sections of the rod.



Consider the segment C of the rod between x and $x+\Delta x$. The heat leaving C per unit time is

$$(1) \qquad \iint_{\partial C} \vec{v}\cdot\vec{n}\, dS$$

where $\vec{v}\cdot\vec{n}$ is the component of the heat velocity $\vec{v}$ in the direction of the outward pointing unit normal $\vec{n}$ to the boundary ∂C of C and dS denotes the element of surface area on ∂C. Since $\vec{v} = k_1(u - T_0)$ on the lateral sides of C and $\vec{v} = -k_2 \nabla u$ along the axis of the rod, (1) yields

6pts. x here

$$(2) \qquad \iint_{\partial C} \vec{v}\cdot\vec{n}\, dS = -k_2 u_x(x+\Delta x, t)A + k_2 u_x(x,t)A + \int_x^{x+\Delta x} k_1(u(\xi,t) - T_0)P\, d\xi$$

where A is the cross sectional area of the rod and P is the perimeter of the cross section of the rod.

On the other hand, the total amount of heat in C at time t is

$$(3) \qquad H(t) = \iiint_C E(x,y,z,u(x,y,z,t))\, dV = \int_x^{x+\Delta x} E(\xi, u(\xi,t))A\, d\xi$$

where $E(x,u)$ denotes the heat energy density at position x in the rod and

(OVER)

Math 325

Summer 2006

Exam I

$n = 15$

$\mu = 71.8$

$\sigma = 13.5$

Distribution of Scores:

	Grad	Undergrad	Frequency
87 – 100	A	A	3
73 – 86	B	B	2
60 – 72	C	B	8
50 – 59	C	C	2
0 – 49	F	D	0