Mathematics 325 | Exam II Fall 2005 | Name: Dr. Grow

1.(25 pts.) Solve $u_t - u_{xx} = 0$ in $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition

$$u(x,0) = \phi(x) = \begin{cases} 1 & \text{if } x > 0, \\ 3 & \text{if } x < 0. \end{cases}$$

Express your answer in terms of the error function:

$$Erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-p^2}\, dp.$$

6 pts. to here

12 pts. to here.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} 3\, dy + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} 1\, dy$$

Let $p = \frac{y-x}{\sqrt{4t}}$. Then $dp = \frac{dy}{\sqrt{4t}}$. $y = 0 \Rightarrow p = \frac{-x}{\sqrt{4t}}$. $y \to \pm\infty \Rightarrow p \to \pm\infty$

$$\therefore\ u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{4t}}} e^{-p^2} 3\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4t}}}^{\infty} e^{-p^2} dp.$$ 16 pts. to here.

$$\int_{-\infty}^{w} e^{-p^2} dp = \int_{-\infty}^{0} e^{-p^2} dp + \int_0^w e^{-p^2} dp = \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} Erf(w) = \frac{\sqrt{\pi}}{2}(1 + Erf(w))$$

$$\int_{w}^{\infty} e^{-p^2} dp = \int_0^{\infty} e^{-p^2} dp - \int_0^w e^{-p^2} dp = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} Erf(w) = \frac{\sqrt{\pi}}{2}(1 - Erf(w)).$$

$$\therefore\ u(x,t) = \frac{3}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\left(1 + Erf\left(\frac{-x}{\sqrt{4t}}\right)\right) + \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\left(1 - Erf\left(\frac{-x}{\sqrt{4t}}\right)\right)$$ 22 pts. to here.

But the error function is clearly odd (i.e. $Erf(-w) = -Erf(w)$) so

$$u(x,t) = \frac{3}{2}\left(1 - Erf\left(\frac{x}{\sqrt{4t}}\right)\right) + \frac{1}{2}\left(1 + Erf\left(\frac{x}{\sqrt{4t}}\right)\right)$$

$$\boxed{u(x,t) = 2 - Erf\left(\frac{x}{\sqrt{4t}}\right)}$$ 25 pts. to here.

2.(25 pts.) Solve $u_{tt} - 4u_{xx} = 0$ in $-\infty < x < \infty$, $-\infty < t < \infty$, subject to the initial conditions $u(x,0) = \ln(1+x^2)$ and $u_t(x,0) = 4 + x$ for $-\infty < x < \infty$.

8 pts. to here. $$u(x,t) = \frac{1}{2}\left[\varphi(x-2t) + \varphi(x+2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \psi(\xi)\,d\xi$$ (d'Alembert's formula)

16 pts. to here. $$= \frac{1}{2}\left[\ln\left[1+(x-2t)^2\right] + \ln\left[1+(x+2t)^2\right]\right] + \frac{1}{4}\int_{x-2t}^{x+2t}(4+\xi)\,d\xi$$

$$= \frac{1}{2}\ln\left\{\left[1+(x-2t)^2\right]\left[1+(x+2t)^2\right]\right\} + \frac{1}{4}\left(4\xi + \frac{\xi^2}{2}\right)\Bigg|_{x-2t}^{x+2t}$$

$$= \frac{1}{2}\ln\left\{\left[1+(x-2t)^2\right]\left[1+(x+2t)^2\right]\right\} + \left(\xi + \frac{\xi^2}{8}\right)\Bigg|_{x-2t}^{x+2t}$$

$$= \frac{1}{2}\ln\left\{\left[1+(x-2t)^2\right]\left[1+(x+2t)^2\right]\right\} + (x+2t)-(x-2t) + \frac{(x+2t)^2-(x-2t)^2}{8}$$

25 pts. to here. $$\boxed{u(x,t) = \frac{1}{2}\ln\left\{\left[1+(x-2t)^2\right]\left[1+(x+2t)^2\right]\right\} + 4t + xt}$$

3.(24 pts.) (a) State, **BUT DO NOT PROVE,** the maximum principle for solutions to the diffusion equation.

(b) Give a physical interpretation of the maximum principle in the case of heat conduction in a thin rod with its lateral sides insulated.

(c) Show, by exhibiting a counterexample, that solutions to the wave equation need not satisfy a maximum principle.

(a) Let $u = u(x,t)$ be a solution to $u_t - k u_{xx} = 0$ in $R: 0 < x < L,\ 0 < t \leq T$, and let u be continuous in $R \cup \partial R$: $0 \leq x \leq L,\ 0 \leq t \leq T$. Then the maximum value of u in $R \cup \partial R$ is attained either on the initial wall (i.e. when $t=0$) or on a sidewall (i.e. when $x=0$ or $x=L$).

8 pts.

(b) The hottest temperature of such a thin rod occurs eithers initially (i.e. when $t=0$) or at one of the ends of the rod (i.e. when $x=0$ or $x=L$).

8 pts.

(c) $u(x,t) = \sin(x)\sin(ct)$ solves the wave equation $u_{tt} - c^2 u_{xx} = 0$ in $\overline{R}: 0 \leq x \leq \pi,\ 0 \leq t \leq \frac{\pi}{c}$, but the maximum value of u on $\overline{R}$ does not occur at a boundary point of $\overline{R}$ but rather at the "center" point $(\frac{\pi}{2}, \frac{\pi}{2c})$ of $\overline{R}$.

8 pts.

4.(26 pts.) Use the Fourier transform method to find a formula for the solution to the inhomogeneous diffusion problem in the upper half-plane:

$$\text{①} \quad u_t - ku_{xx} = f(x,t) \quad \text{if } -\infty < x < \infty,\ 0 < t < \infty,$$

$$\text{②} \quad u(x,0) = \phi(x) \quad \text{if } -\infty < x < \infty.$$

Notes: 1. The solution is rumored to be

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(y,s) \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} dy ds + \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} dy.$$

2. If you cannot solve this problem in its full generality, to earn half the points, do the special case when $f(x,t) \equiv 0$.

We take the Fourier transform of ① with respect to the variable x:

$$\mathcal{F}(u_t - ku_{xx})(\xi) = \mathcal{F}(f(\cdot,t))(\xi)$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - k(i\xi)^2 \mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

3 pts. to here

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 \mathcal{F}(u)(\xi) = \hat{f}(\xi,t) \quad \leftarrow$$ Linear first-order ODE in the variable t for each fixed ξ.

5 pts. here

Integrating factor: $e^{\int k\xi^2 dt} = e^{k\xi^2 t}$.

$$e^{k\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}(u)(\xi) = e^{k\xi^2 t}\hat{f}(\xi,t)$$

$$\frac{\partial}{\partial t}\left[e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right] = e^{k\xi^2 t}\hat{f}(\xi,t).$$

Integrating with respect to t (holding ξ fixed) yields:

5 pts. to here.

$$e^{k\xi^2 t}\mathcal{F}(u)(\xi) = \int_0^t \hat{f}(\xi,\tau)e^{k\xi^2\tau}\,d\tau + c(\xi)$$

$$\mathcal{F}(u)(\xi) = e^{-k\xi^2 t}\int_0^t \hat{f}(\xi,\tau)e^{k\xi^2\tau}\,d\tau + e^{-k\xi^2 t}c(\xi)$$

10 — to here.

$$= \int_0^t \hat{f}(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau + e^{-k\xi^2 t}c(\xi).$$

Applying the initial condition ② produces

$$\hat{\varphi}(\xi) = \mathcal{F}\big(u(\cdot,0)\big)(\xi) = 0 + c(\xi)e^{0} = c(\xi).$$

13 pts. to here. $$\therefore \quad \mathcal{F}(u)(\xi) = \int_0^t \hat{f}(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau + \hat{\varphi}(\xi)e^{-k\xi^2 t}$$

Apply table entry I: $\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{1}{\sqrt{2a}}e^{-\xi^2/4a}$ *with* $a = \frac{1}{4kt}$

to obtain $\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi) = e^{-k\xi^2 t}$ *and hence*

17 pts. to here. $$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,d\tau + \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi).$$

But $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = \mathcal{F}\left(\frac{f*g}{\sqrt{2\pi}}\right)(\xi)$ *so*

21 pts. to here. $$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\left(f(\cdot,\tau) * \frac{1}{\sqrt{4\pi k(t-\tau)}}e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,d\tau + \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$$

Interchanging the order of integration in the first term on the right side of the equation gives

22 pts. to here. $$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4\pi k(t-\tau)}}e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\,d\tau\right)(\xi) + \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi).$$

Using the inversion formula then gives the desired result:

26 pts. to here. $$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(y,\tau)\frac{1}{\sqrt{4\pi k(t-\tau)}}e^{\frac{-(x-y)^2}{4k(t-\tau)}}\,dy\,d\tau + \int_{-\infty}^{\infty} \varphi(y)\frac{1}{\sqrt{4k\pi t}}e^{-\frac{(x-y)^2}{4kt}}\,dy.}$$

Grading key for $f(x,t) \equiv 0$ case of #4.

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 \mathcal{F}(u)(\xi) = 0$$ 2 pts. to here.

$$\frac{\partial}{\partial t}\left[e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right] = 0$$ 3

$$\mathcal{F}(u)(\xi) = e^{-k\xi^2 t} c(\xi)$$ 4

$$\hat{\varphi}(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = e^{0} c(\xi) = c(\xi)$$ 5

$$\mathcal{F}(u)(\xi) = e^{-k\xi^2 t}\hat{\varphi}(\xi)$$ 6

Table entry I with $a = \frac{1}{4kt}$

$$\mathcal{F}(u)(\xi) \stackrel{\swarrow}{=} \mathcal{F}(\varphi)(\xi)\, \mathcal{F}\left(\frac{1}{\sqrt{2kt}} e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$$ 8

$$= \frac{1}{\sqrt{2\pi}} \mathcal{F}\left(\varphi * \frac{1}{\sqrt{2kt}} e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$$ 10

$$= \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$$ 11

$$\therefore\ u(x,t) = \left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}}\right)(x)$$ 12

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} \varphi(y) \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}\, dy}$$ 13 pts. to here

Math 325

Exam II

Fall 2005

Mean: 70.8

Standard Deviation: 20.1

Distribution of Scores:

	Grad. Letter Grade	Undergrad. Letter Grade	Frequency
87–100	A	A	3
73 – 86	B	B	8
60 – 72	C	B	4
50 – 59	C	C	1
0 – 49	F	D	3