Mathematics 325 Exam II Summer 2002 Name: Dr. Grow

1. (25 pts.) Solve
$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$
subject to the initial condition
$$u(x,0) = \phi(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$
Express your answer in terms of the error function:
$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2}\,dp .$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \cdot e^{-y}\,dy = \int_0^{\infty} \frac{e^{-\frac{(x-y)^2+4ty}{4t}}}{\sqrt{4\pi t}}\,dy$$

Complete square (in y) in exponent:
$$(x-y)^2 + 4ty = y^2 - 2xy + 4ty + x^2 = y^2 + 2y(2t-x) + (2t-x)^2 + x^2 - (2t-x)^2$$
$$= (y+2t-x)^2 + \cancel{x^2} - 4t^2 + 4tx - \cancel{x^2} = (y+2t-x)^2 + 4tx - 4t^2.$$

$$\therefore\ u(x,t) = \int_0^{\infty} \frac{e^{-\frac{(y+2t-x)^2+4tx-4t^2}{4t}}}{\sqrt{4\pi t}}\,dy = e^{t-x} \cdot \underbrace{\int_0^{\infty} \frac{e^{-\frac{(y+2t-x)^2}{4t}}}{\sqrt{4\pi t}}\,dy}_{\text{Let } p = \frac{y+2t-x}{\sqrt{4t}}.\ \text{Then } dp = \frac{dy}{\sqrt{4t}}.}$$

$$\therefore\ u(x,t) = \frac{e^{t-x}}{\sqrt{\pi}} \cdot \int_{\frac{2t-x}{\sqrt{4t}}}^{\infty} e^{-p^2}\,dp.$$

But
$$\int_w^{\infty} e^{-p^2}dp = \left(\int_0^{\infty} - \int_0^{w}\right) e^{-p^2}dp = \frac{\sqrt{\pi}}{2} - \int_0^{w} e^{-p^2}dp = \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{Erf}(w)\right).$$

$$\therefore\ u(x,t) = \frac{e^{t-x}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{Erf}\left(\frac{2t-x}{\sqrt{4t}}\right)\right) = \boxed{\frac{e^{t-x}}{2}\left(1 - \mathrm{Erf}\left(\frac{2t-x}{\sqrt{4t}}\right)\right)}.$$

2. (25 pts.) (a) By exhibiting an appropriate function defined on an appropriate region, show that solutions to the wave equation need not satisfy a maximum principle.

(b) State without proof the theorem known as the weak maximum principle for solutions to the diffusion equation.

(c) State without proof the theorem known as the weak maximum principle for solutions to Laplace's equation.

Bonus (10 pts.): Give a mathematical proof for the theorem stated in either part (b) or part (c) above.

(a) $\boxed{u(x,t) = \sin(\pi x)\sin(\pi t)}$ solves $u_{tt} - u_{xx} = 0$ in $\boxed{R: 0<x<1,\ 0<t<1,}$

yet $\max\limits_{(x,t)\in\overline{R}\setminus R} u(x,t) = 0 < 1 = u(\tfrac{1}{2},\tfrac{1}{2}) = \max\limits_{(x,t)\in\overline{R}} u(x,t)$.

(b) Let $u = u(x,t)$ be a solution to the diffusion equation $u_t - ku_{xx} = 0$ in $R: 0<x<\ell,\ 0<t\le T$, and let u be continuous on $\overline{R}: 0\le x\le \ell,\ 0\le t\le T$.

Then $\max\limits_{(x,t)\in\overline{R}} u(x,t) = \max\limits_{(x,t)\in\overline{R}\setminus R} u(x,t)$.

(c) Let $u = u(x,y)$ be a solution to Laplace's equation $u_{xx} + u_{yy} = 0$ in an open bounded domain D in the xy-plane, and let u be continuous on $\overline{D} = D \cup \partial D$.

Then $\max\limits_{(x,y)\in\overline{D}} u(x,y) = \max\limits_{(x,y)\in\partial D} u(x,y)$.

3. (25 pts.) Let g be an absolutely integrable function on $(-\infty,\infty)$. Use Fourier transform methods to solve

① $u_{xx} + u_{yy} = 0$ for $-\infty < x < \infty$, $0 < y < \infty$,

subject to the boundary condition

② $$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise}, \end{cases} = \chi_{(-1,1)}(x)$$

and the decay conditions

③ $\lim_{y\to\infty} u(x,y) = 0$ for each x in $(-\infty,\infty)$

and ④ $|u(x,y)| \le |g(x)|$ for all x in $(-\infty,\infty)$ and all $y > 0$.

$$\mathcal{F}(u_{xx}+u_{yy})(\xi) \overset{①}{=} \mathcal{F}(0)(\xi) \Rightarrow (i\xi)^2\mathcal{F}(u)(\xi) + \frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) - \xi^2\mathcal{F}(u)(\xi) = 0 \Rightarrow \mathcal{F}(u)(\xi) = A(\xi)e^{-\xi y} + B(\xi)e^{\xi y}.$$

If $\xi > 0$ then $0 \overset{③④}{=} \lim_{y\to\infty}\mathcal{F}(u(\cdot,y))(\xi) = \lim_{y\to\infty}\left[A(\xi)e^{-\xi y} + B(\xi)e^{\xi y}\right] = \lim_{y\to\infty} B(\xi)e^{\xi y}$

so $B(\xi) = 0$. A similar argument shows $A(\xi) = 0$ if $\xi < 0$. Thus

12 $$\mathcal{F}(u)(\xi) = \begin{cases} A(\xi)e^{-\xi y} & \text{if } \xi > 0, \\ B(\xi)e^{\xi y} & \text{if } \xi < 0, \end{cases} = C(\xi)e^{-|\xi|y}.$$

$\therefore \mathcal{F}(\chi_{(-1,1)})(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = C(\xi)e^{-|\xi|0} = C(\xi)$. Also, formula C in the table of Fourier transforms, $\mathcal{F}\left(\frac{1}{(\cdot)^2+a^2}\right)(\xi) = \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|\xi|}}{a}$,

with $a = y$ gives $\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi) = e^{-|\xi|y}$.

$$\therefore \mathcal{F}(u)(\xi) = C(\xi)e^{-y|\xi|} = \mathcal{F}(\chi_{(-1,1)})(\xi)\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\chi_{(-1,1)} * \sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

18 $= \mathcal{F}\left(\frac{1}{\pi}\frac{y}{(\cdot)^2+y^2} * \chi_{(-1,1)}\right)(\xi)$. By the inversion theorem,

22 $$u(x,y) = \frac{1}{\pi}\left(\frac{y}{(\cdot)^2+y^2} * \chi_{(-1,1)}\right)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\chi_{(-1,1)}(z)\cdot\frac{y}{(x-z)^2+y^2}\,dz = \frac{1}{\pi}\int_{-1}^{1}\frac{y\,dz}{(x-z)^2+y^2}$$

(OVER)

$$u(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\left(\frac{z-x}{y}\right)^2+1}\cdot\frac{dz}{y}\,.\qquad \text{Let}\quad w = \frac{z-x}{y}\,.\ \text{Then}\ dw = \frac{dz}{y}\,.$$

$$= \frac{1}{\pi}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}} \frac{1}{w^2+1}\,dw$$

$$= \frac{1}{\pi}\operatorname{Arctan}\left(\frac{1-x}{y}\right) - \frac{1}{\pi}\operatorname{Arctan}\left(\frac{-1-x}{y}\right)$$

Using the fact that Arctangent is an odd function (i.e. $\operatorname{Arctan}(-z) = -\operatorname{Arctan}(z)$) it follows that

$$\boxed{u(x,y) = \frac{1}{\pi}\operatorname{Arctan}\left(\frac{x+1}{y}\right) - \frac{1}{\pi}\operatorname{Arct}\left(\frac{x-1}{y}\right)}$$

4. (25 pts.) Find a solution to

① $u_t - u_{xx} = 0$ for $0 < x < \pi$, $0 < t < \infty$,

subject to the MIXED boundary conditions

②-③ $u(0,t) = 0 = u_x(\pi,t)$ for $t \ge 0$

(Dirichlet at the left, Neumann at the right), and the initial condition

④ $u(x,0) = 3\sin(x/2) - 2\sin(3x/2)$ for $0 \le x \le \pi$.

Bonus (10 pts.): Show that there is only one solution to the problem above.

(Via separation of variables) Let $u(x,t) = X(x)T(t)$ be a nontrivial solution of ①-②-③. Then

$$X(x)T'(t) = X''(x)T(t) \Rightarrow \frac{-T'(t)}{T(t)} = \frac{-X''(x)}{X(x)} = \text{constant} = \lambda.$$ Applying ②-③

gives $X(0)T(t) = 0$ and $X'(\pi)T(t) = 0$ for all $t \ge 0$. Thus

6 $$\begin{cases} X''(x) + \lambda X(x) = 0, & X(0) = 0 = X'(\pi) \\ T'(t) + \lambda T(t) = 0 \end{cases}$$

Case $\lambda > 0$ (say $\lambda = \beta^2$ where $\beta > 0$): Then $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x) \Rightarrow X'(x) = -\beta c_1\sin(\beta x) + \beta c_2\cos(\beta x)$

$0 = X(0) = c_1$ and $0 = X'(\pi) = -\beta \cancel{c_1}^{0}\sin(\beta\pi) + \beta c_2\cos(\beta\pi) \Rightarrow \cos(\beta\pi) = 0 \Rightarrow \beta = (2n+1)\frac{1}{2}$

where $n = 0, 1, 2, \ldots$ Eigenvalues: $\lambda_n = (n+\frac{1}{2})^2$ Eigenfunctions: $X_n(x) = \sin((n+\frac{1}{2})x)$.

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Case $\lambda = 0$: $X(x) = c_1x + c_2 \Rightarrow X'(x) = c_1$. $0 = X(0) = c_2$ and $0 = X'(\pi) = c_1$ Trivial solutions.

Case $\lambda < 0$ (say $\lambda = -\beta^2$ where $\beta > 0$): $X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x) \Rightarrow X'(x) = \beta c_1\sinh(\beta x) + \beta c_2\cosh(\beta x)$

$0 = X(0) = c_1$ and $0 = X'(\pi) = \beta \cancel{c_1}^{0}\sinh(\beta\pi) + \beta c_2\cosh(\beta\pi) \Rightarrow c_2 = 0$ Trivial solutions.

20 $T_n'(t) + \lambda_n T_n(t) = 0 \Rightarrow T_n(t) = e^{-\lambda_n t} = e^{-(n+\frac{1}{2})^2 t}$ (up to a constant factor).

$\therefore u_n(x,t) = X_n(x)T_n(t) = \sin((n+\frac{1}{2})x)e^{-(n+\frac{1}{2})^2 t}$ solves ①-②-③ for $n = 0, 1, 2, \ldots$

23 The superposition principle implies $u(x,t) = \sum_{n=0}^{N} a_n \sin((n+\frac{1}{2})x)e^{-(n+\frac{1}{2})^2 t}$ solves ①-②-③

for all integers $N \ge 1$ and any choice of constants $a_0, \ldots, a_N$. For all $0 \le x \le \pi$ we want

24 ④: $3\sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right) = u(x,0) = a_0\sin\left(\frac{x}{2}\right)e^{-(\frac{1}{2})^2 0} + a_1\sin\left(\frac{3x}{2}\right)e^{-(\frac{3}{2})^2 0} + \ldots + a_N\sin\left((N+\frac{1}{2})x\right)e^{-(N+\frac{1}{2})^2 0}$.

By inspection, $3 = a_0$, $-2 = a_1$, and all other $a_n = 0$.

25 $$\therefore \boxed{u(x,t) = 3\sin\left(\frac{x}{2}\right)e^{-t/4} - 2\sin\left(\frac{3x}{2}\right)e^{-9t/4}}$$

(OVER for Bonus)

<u>Bonus</u>: Let $v = v(x,t)$ be any other solution to ①-②-③-④ and consider the energy function

$$E(t) = \int_0^{\pi} w^2(x,t)\,dx \qquad (t \geq 0)$$

of the difference $w(x,t) = u(x,t) - v(x,t)$. Then

$$\frac{dE}{dt} = \int_0^{\pi} \frac{\partial}{\partial t}\left[w^2(x,t)\right] dx = \int_0^{\pi} 2w(x,t)\, w_t(x,t)\,dx.$$

Using ① we find that

$$\frac{dE}{dt} = \int_0^{\pi} \overbrace{2w(x,t)}^{U}\, \overbrace{w_{xx}(x,t)\,dx}^{dV}$$

$$= 2w(x,t)\,w_x(x,t)\Big|_{x=0}^{\pi} - 2\int_0^{\pi} w_x^2(x,t)\,dx.$$

Applying ②-③ we find that $2w(\pi,t)\overbrace{w_x(\pi,t)}^{0} - 2\overbrace{w(0,t)}^{0}w_x(0,t) = 0$ so $\frac{dE}{dt} \leq 0$ for all $t \geq 0$. Thus $0 \leq E(t) \leq E(0)$ for all $t \geq 0$. But $w(x,0) = u(x,0) - v(x,0) = 0$ for all $0 \leq x \leq \pi$ by ④ so $E(0) = 0$. The vanishing theorem implies $w^2(x,t) = 0$ for all $0 \leq x \leq \pi$ and all $t \geq 0$; i.e. $u(x,t) = v(x,t)$ for all $0 \leq x \leq \pi$ and all $t \geq 0$. (This completes the proof of uniqueness of solutions to ①-②-③-④.)