Mathematics 325 | Exam II Summer 2000 | Name: Dr. Grow

1\. (25 pts.) (a) State and prove the theorem known as the (weak) maximum-minimum principle for (solutions to) Laplace's equation.
(b) Give a physical interpretation for the result in part (a).

(a) __Theorem__: If $u=u(x,y)$ solves $u_{xx}+u_{yy}=0$ in an open bounded set $D$ in the plane and is continuous on the closure $\bar{D}=D\cup\partial D$ of $D$, then the maximum and minimum values of $u$ on $\bar{D}$ are attained on the boundary $\partial D$ of $D$.

__Proof__: Let $\varepsilon>0$ and let $v(x,y)=u(x,y)+\varepsilon(x^2+y^2)$ for $(x,y)$ in $\bar{D}$. We claim that $v$ attains its maximum on $\bar{D}$ at a point of $\partial D$. For suppose not; then $v$ attains it maximum on $\bar{D}$ at (an interior point) $(x_0,y_0)$ in $D$. This implies $v_x(x_0,y_0)=v_y(x_0,y_0)=0$, $v_{xx}(x_0,y_0)\le 0$, and $v_{yy}(x_0,y_0)\le 0$. But then

$$0\ge v_{xx}(x_0,y_0)+v_{yy}(x_0,y_0)=u_{xx}(x_0,y_0)+u_{yy}(x_0,y_0)+4\varepsilon=4\varepsilon,$$

which contradicts $\varepsilon>0$. This proves the claim.

Note that $M=\max\limits_{(x,y)\in\partial D}(x^2+y^2)$ is finite since $D$ is bounded. Thus

$$(\dagger)\quad \max_{(x,y)\in\bar{D}} u(x,y)\le \max_{(x,y)\in\bar{D}}\left[u(x,y)+\varepsilon(x^2+y^2)\right]=\max_{(x,y)\in\bar{D}} v(x,y)=\max_{(x,y)\in\partial D} v(x,y)$$

$$=\max_{(x,y)\in\partial D}\left[u(x,y)+\varepsilon(x^2+y^2)\right]\le \varepsilon M+\max_{(x,y)\in\partial D} u(x,y).$$

Because $\varepsilon>0$ is abitrary, it follows from $(\dagger)$ that $\max\limits_{(x,y)\in\bar{D}} u(x,y)\le\max\limits_{(x,y)\in\partial D} u(x,y)$. Since $\partial D\subseteq\bar{D}$, the reverse inequality clearly holds, and hence $\max\limits_{(x,y)\in\bar{D}} u(x,y)=\max\limits_{(x,y)\in\partial D} u(x,y)$; i.e. the maximum value of $u$ on the closure of $D$ is attained on the boundary of $D$.

To prove the analogous minimum value result, note that $-u$ solves the Laplace equation in $D$ and is continuous on $\bar{D}$. Thus, the maximum principle holds for $-u$, so

$$\min_{(x,y)\in\bar{D}} u(x,y)=-\max_{(x,y)\in\bar{D}}\left[-u(x,y)\right]=-\max_{(x,y)\in\partial D}\left[-u(x,y)\right]=\min_{(x,y)\in\partial D} u(x,y).$$

[See back side for (b).]

#1 (b) The steady-state temperature $u(x,y)$ at position $(x,y)$ of a laminar region $\bar{D}$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ in $D$. If $D$ is bounded then the maximum-minimum principle says that the hottest and coldest temperatures in $\bar{D}$ occur on the boundary of $D$.

2. (25 pts.) Find the solution to

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

which satisfies

$$u(x,0) = x^2 \quad \text{for } -\infty < x < \infty.$$

You may find the following identities useful:

$$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} pe^{-p^2}\,dp = 0, \qquad \int_{-\infty}^{\infty} p^2e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}.$$

By a formula from Section 2.4, a candidate for the solution is

$$(*) \qquad u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\,y^2\,dy \qquad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty.$$

[Since $\varphi(y) = y^2$ is not bounded on $-\infty < y < \infty$, we are not guaranteed that this formula actually gives the solution to the problem; we will need to check our final form.]

Make the substitution $p = \frac{y-x}{\sqrt{4t}}$ in the integral of $(*)$. Then $dp = \frac{dy}{\sqrt{4t}}$ so

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(x + p\sqrt{4t})^2\,dp$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left(x^2 + 2xp\sqrt{4t} + 4tp^2\right)dp$$

$$= \frac{x^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{2x\sqrt{4t}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}p\,dp + \frac{4t}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}p^2\,dp$$

$$= x^2\cdot 1 + \frac{2x\sqrt{4t}}{\sqrt{\pi}}\cdot 0 + 4t\cdot\frac{1}{2}$$

$$\boxed{u(x,t) = x^2 + 2t}$$

Check: $u_t - u_{xx} = 2 - 2 \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^2$

3.(25 pts.) Let $f$ and $\psi$ be piecewise-continuous, absolutely integrable functions on $(-\infty,\infty)$. Use Fourier transform methods to solve

$$u_{xx} + u_{yy} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the boundary condition

$$u(x,0) = f(x) \quad \text{for } -\infty < x < \infty$$

and the decay conditions

$$\lim_{y\to\infty} u(x,y) = 0 \quad \text{for each } x \in (-\infty,\infty)$$

and, for each $y > 0$,

$$|u(x,y)| \le |\psi(x)| \quad \text{for all } -\infty < x < \infty.$$

Bonus(10 pts.): Compute an explicit formula for the solution to the above problem if the function $f$ is given by $f(x) = 1$ for $|x| < 1$, and $f(x) = 0$ otherwise.

$$\mathcal{F}(u_{xx}+u_{yy})(\xi) = \mathcal{F}(0)(\xi)$$

$$\mathcal{F}(u_{xx})(\xi) + \mathcal{F}(u_{yy})(\xi) = 0$$

$$(i\xi)^2\mathcal{F}(u)(\xi) + \frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) - \xi^2\mathcal{F}(u)(\xi) = 0$$

(*) $$\mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}$$

The decay conditions imply

(**) $$\lim_{y\to\infty}\mathcal{F}(u(\cdot,y))(\xi) = \lim_{y\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,y)e^{-i\xi x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\lim_{y\to\infty}u(x,y)e^{-i\xi x}\,dx$$

$$= 0$$

for all $-\infty < \xi < \infty$. On the other hand,

(***) $$\begin{cases}\lim\limits_{y\to\infty} e^{\xi y} = +\infty & \text{if } \xi > 0,\\ \lim\limits_{y\to\infty} e^{-\xi y} = +\infty & \text{if } \xi < 0.\end{cases}$$

Therefore (*)-(**)-(***) imply $c_1(\xi)=0$ if $\xi>0$ and $c_2(\xi)=0$ if $\xi<0$. Thus

$$\mathcal{F}(u)(\xi) = \left\{\begin{array}{ll} c_2(\xi)e^{-\xi y} & \text{if } \xi>0 \\ c_1(\xi)e^{\xi y} & \text{if } \xi<0\end{array}\right\} = c(\xi)e^{-|\xi|y}.$$

Applying the boundary condition yields

$$\mathcal{F}(f)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)e^{0} = c(\xi).$$

Thus,

$$\mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)e^{-|\xi|y} \overset{\text{Table, Entry C (with } a=y)}{=} \mathcal{F}(f)(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f * \sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

Therefore

$$u(x,y) = \frac{1}{\pi}\left(f * \frac{y}{(\cdot)^2+y^2}\right)(x)$$

$$\boxed{u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y f(s)\,ds}{(x-s)^2+y^2}}$$

for $-\infty<x<\infty$, $0<y<\infty$.

(OVER FOR BONUS)

#3 <u>BONUS</u>: If $f(x) = \begin{cases} 1 & \text{for } |x| < 1, \\ 0 & \text{otherwise}, \end{cases}$

then the solution to #3 is

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(s)\,y\,ds}{(x-s)^2+y^2} = \frac{1}{\pi}\int_{-1}^{1} \frac{1\cdot y\,ds}{(x-s)^2+y^2}$$

Let $v = \frac{s-x}{y}$

Then $dv = \frac{ds}{y}$

$$\therefore \quad u(x,y) = \frac{1}{\pi}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}} \frac{y^2\,dv}{y^2v^2+y^2}$$

$$= \frac{1}{\pi}\int_{\frac{-(1+x)}{y}}^{\frac{1-x}{y}} \frac{dv}{v^2+1}$$

$$= \frac{1}{\pi}\left[\operatorname{Arctan}\left(\frac{1-x}{y}\right) - \operatorname{Arctan}\left(-\frac{(1+x)}{y}\right)\right]$$

$$\boxed{u(x,y) = \frac{1}{\pi}\left[\operatorname{Arctan}\left(\frac{x+1}{y}\right) - \operatorname{Arctan}\left(\frac{x-1}{y}\right)\right]} \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty.$$

4. (25 pts.) Use separation of variables to find a formal solution to

$$u_t - u_{xx} = 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

subject to

$$u_x(0,t) = 0 = u_x(1,t) - u(1,t) \quad \text{for } 0 \le t < \infty,$$

and

$$u(x,0) = \phi(x) \quad \text{for } 0 \le x \le 1$$

where $\phi$ is a (given) continuous function on [0,1].
Bonus (10 pts.): Is there at most one solution to the above problem? Why or why not?

$$u(x,t) = X(x)T(t) \Rightarrow X(x)T'(t) - X''(x)T(t) = 0 \Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \text{const.} = -\lambda .$$

$$X'(0)T(t) = 0 \text{ for } 0 \le t \Rightarrow X'(0) = 0;\quad X'(1)T(t) - X(1)T(t) = 0 \text{ for } 0 \le t \Rightarrow X'(1) = X(1).$$

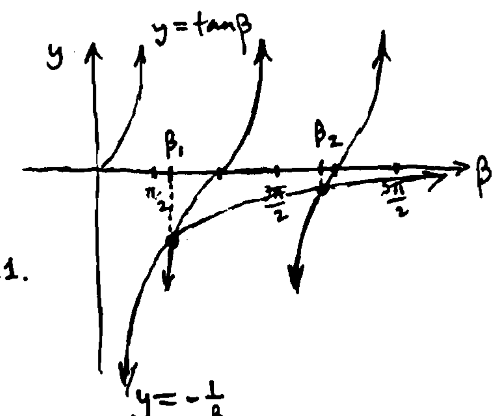
$$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0,\ X'(0) = 0 = X'(1) - X(1)} \text{ (eigenvalue problem)} \\ T'(t) + \lambda T(t) = 0 \end{cases}$$

<u>Case $\lambda > 0$ (say $\lambda = \beta^2$ where $\beta > 0$):</u>

$X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$ $\qquad 0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$

$X'(x) = -\beta c_1\sin(\beta x) + \beta c_2\cos(\beta x)$ $\qquad 0 = X'(1) - X(1) = [-\beta\sin(\beta) - \cos(\beta)]c_1$ (nonzero)

$\Rightarrow \tan(\beta) = -\frac{1}{\beta}$ $\quad (*)$




igure1.

There is an infinite sequence of positive eigenvalues $\lambda_1 = \beta_1^2$, $\lambda_2 = \beta_2^2$, ... where $\beta_n$ is the $n^{th}$ positive solution to $(*)$. (See figure 1.)

Eigenfunctions: $X_n(x) = \cos(\beta_n x)$ $\quad (n = 1, 2, 3, \dots)$

$$\therefore T_n'(t) + \lambda_n T_n(t) = 0 \Rightarrow T_n(t) = e^{-\lambda_n t} = e^{-\beta_n^2 t}$$

$(\beta_1 \approx 2.79838604578,\ \beta_2 \approx 6.1212504669, \dots)$

<u>Case $\lambda = 0$:</u>

$X(x) = c_1 x + c_2$ $\qquad 0 = X'(0) = c_1$

$X'(x) = c_1$ $\qquad 0 = X'(1) - X(1) = 0 - c_2$

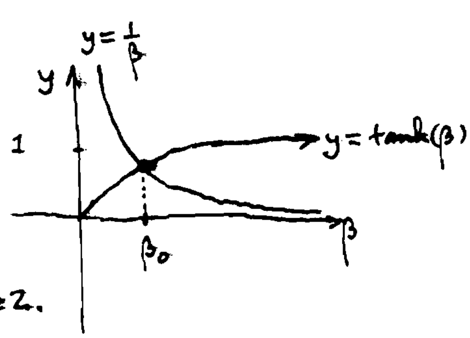
} No nontrivial solutions in this case.

<u>Case $\lambda < 0$ (say $\lambda = -\beta^2$ where $\beta > 0$):</u>

$X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x)$ $\qquad 0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$

$X'(x) = \beta c_1\sinh(\beta x) + \beta c_2\cosh(\beta x)$ $\qquad 0 = X'(1) - X(1) = [\beta\sinh(\beta) - \cosh(\beta)]c_1$ (nonzero)

$\Rightarrow \tanh(\beta) = \frac{1}{\beta}$ $\quad (**)$




igure 2.

There is one negative eigenvalue $\lambda_0 = -\beta_0^2$ where $\beta_0$ is the unique positive solution to $(**)$. (See figure 2.) $(\beta_0 \approx 1.19967864026)$

Eigenfunction: $X_0(x) = \cosh(\beta_0 x)$

$$T_0'(t) + \lambda_0 T_0(t) = 0 \Rightarrow T_0(t) = e^{-\lambda_0 t} = e^{\beta_0^2 t}$$

(cont.)

#4 (cont.)

$$\therefore \quad \boxed{u(x,t) = c_0 \cosh(\beta_0 x) e^{\beta_0^2 t} + \sum_{n=1}^{\infty} c_n \cos(\beta_n x) e^{-\beta_n^2 t}}$$

($c_0, c_1, c_2, \ldots$ "arbitrary" constants)

is a formal solution to the homogeneous part of #4. In order to satisfy the initial condition the constants must be chosen (if possible) to satisfy

$$\boxed{\varphi(x) = u(x,0) = c_0 \cosh(\beta_0 x) + \sum_{n=1}^{\infty} c_n \cos(\beta_n x) \qquad \text{for all } 0 \le x \le 1.}$$