Mathematics 325 Exam II Name: Dr. Grow

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1.(25 pts.) Solve $u_t - u_{xx} = 0$ $(k=1)$ in $-\infty < x < \infty$, $0 < t < \infty$, subject to $u(x,0) = x^3$ for $-\infty < x < \infty$.

Note: You may find useful the following formulas.

$$\int_{-\infty}^{\infty} p e^{-p^2}\,dp = 0 = \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp, \quad 2\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-p^2}\,dp.$$

4 pts. to here

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}\varphi(y)\,dy = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} y^3\,dy$$

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Let $p = \dfrac{y-x}{\sqrt{4t}}$. Then $dp = \dfrac{dy}{\sqrt{4t}}$

8

12 pts. to here

$$= \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}}\left(x + p\sqrt{4t}\right)^3 dp$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

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$$= \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}}\left(x^3 + 3x^2 p\sqrt{4t} + 3xp^2(4t) + p^3 4t\sqrt{4t}\right)dp$$

20 to here.

$$= x^3 \cancel{\int_{-\infty}^{\infty} e^{-p^2}\frac{dp}{\sqrt{\pi}}}^{\,1} + 3x^2\sqrt{4t}\cancel{\int_{-\infty}^{\infty} pe^{-p^2}\frac{dp}{\sqrt{\pi}}}^{\,0} + 12xt\cancel{\int_{-\infty}^{\infty} e^{-p^2}p^2\frac{dp}{\sqrt{\pi}}}^{\,\frac{1}{2}} + 8t\sqrt{t}\cancel{\int_{-\infty}^{\infty} e^{-p^2}p^3\frac{dp}{\sqrt{\pi}}}^{\,0}$$

4

$$\boxed{u(x,t) = x^3 + 6xt}$$

25

Because $\varphi(x) = x^3$ is unbounded, we need to check our answer.

$u_t = 6x$, $u_{xx} = 6x$, $\therefore\ u_t - u_{xx} = 0$ ✓

$u(x,0) = x^3 + 6x(0) = x^3$ ✓

2.(25 pts.) (a) State without proof the weak maximum principle for solutions to the diffusion equation.
(b) Use part (a) to show that there is at most one solution to

$$u_t - u_{xx} = x(1-x)\cos^2(\pi t) \quad \text{in} \quad 0 < x < 1,\ 0 < t \le 5,$$

subject to the boundary conditions

$$u(0,t) = t \quad \text{and} \quad u(1,t) = 4\cos(2\pi t) - 3\sin(2\pi t) \quad \text{if} \quad 0 \le t \le 5,$$

and the initial condition

$$u(x,0) = 4(1-x)^4 \quad \text{if} \quad 0 \le x \le 1.$$

12 (a) Let $u = u(x,t)$ be a solution to $u_t - ku_{xx} = 0$ in $R: 0 < x < L,\ 0 < t \le T$ and let u be continuous on $\overline{R}: 0 \le x \le L,\ 0 \le t \le T$. Then the maximum value of u on $\overline{R}$ is attained either on the initial wall (i.e. $x=0$) or on the lateral walls (i.e. $x=0$ or $x=L$).

13 (b) Suppose that $u = u_1(x,t)$ and $u = u_2(x,t)$ are two solutions of the problem in part (b). Then $v(x,t) = u_1(x,t) - u_2(x,t)$ is a solution of $v_t - v_{xx} = 0$ in $R: 0 < x < 1,\ 0 < t \le 5$ and satisfies $v(0,t) = 0 = v(1,t)$ if $0 \le t \le 5$ and $v(x,0) = 0$ if $0 \le x \le 1$. Therefore the maximum of v on $\overline{R}$ is zero since v vanishes on the intial and lateral walls. I.e. $v(x,t) \le 0$ for all (x,t) in $\overline{R}$ so $u_1(x,t) \le u_2(x,t)$ on $\overline{R}$. A similar argument with $u_2(x,t) - u_1(x,t)$ in place of $u_1(x,t) - u_2(x,t)$ shows that $u_2(x,t) \le u_1(x,t)$ on $\overline{R}$. It follows that $u_1(x,t) = u_2(x,t)$ on $\overline{R}$. That is, there is at most one solution to the problem in part (b).

3.(25 pts.) Use Fourier transform methods to derive a formula for the solution to

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad -\infty < x < \infty, \ -\infty < t < \infty,$$

subject to

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{if} \quad -\infty < x < \infty.$$

BONUS: For 10 extra points, instead solve the inhomogeneous equation $u_{tt} - c^2 u_{xx} = f(x,t)$ in the xt-plane subject to the initial conditions above.

We take the Fourier transform of $u_{tt} - c^2 u_{xx} = f(x,t)$ with respect to x, yielding

$$\mathcal{F}\left(u_{tt} - c^2 u_{xx}\right)(\xi) = \mathcal{F}\left(f(x,t)\right)(\xi) \Rightarrow \frac{\partial^2 \mathcal{F}(u)(\xi)}{\partial t^2} - c^2 (i\xi)^2 \mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

4 $\Rightarrow \dfrac{\partial^2 \mathcal{F}(u)(\xi)}{\partial t^2} + c^2 \xi^2 \mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$. The general solution of this inhomogeneous 2nd order

+1 ODE in the variable t (with parameter ξ) is $\mathcal{F}(u)(\xi) = \hat{u}_h(\xi,t) + \hat{u}_p(\xi,t)$ where

8 $\hat{u}_h(\xi,t) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t)$ is the general solution of the associated homogeneous equation $\dfrac{\partial^2 \mathcal{F}(u)(\xi)}{\partial t^2} + c^2\xi^2 \mathcal{F}(u)(\xi) = 0$ and a particular solution of the inhomogeneous equation is given by variation of parameters: $\hat{u}_p(\xi,t) = v(\xi,t)\cos(c\xi t) + w(\xi,t)\sin(c\xi t)$

+3 with $v(\xi,t) = \displaystyle\int_0^t - \frac{\hat{f}(\xi,\tau)\sin(c\xi\tau)}{c\xi}\,d\tau$ and $w(\xi,t) = \displaystyle\int_0^t \frac{\hat{f}(\xi,\tau)\cos(c\xi\tau)}{c\xi}\,d\tau$.

(Note that the Wronskian of v and w is $W = \begin{vmatrix} \cos(c\xi t) & \sin(c\xi t) \\ -c\xi\sin(c\xi t) & c\xi\cos(c\xi t) \end{vmatrix} = c\xi$.) Thus,

simplifying gives

+5 $$\hat{u}_p(\xi,t) = \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\left[\cos(c\xi\tau)\sin(c\xi t) - \sin(c\xi\tau)\cos(c\xi t)\right]d\tau = \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\sin(c\xi(t-\tau))\,d\tau$$

and consequently,

$$\mathcal{F}(u)(\xi) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\sin(c\xi(t-\tau))\,d\tau.$$

Applying the initial conditions gives

2 $$c_1(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = \mathcal{F}\left(u(x,0)\right)(\xi) = \mathcal{F}(\varphi)(\xi) \quad \text{and} \quad c\xi c_2(\xi) = \mathcal{F}(u_t)(\xi)\Big|_{t=0} = \mathcal{F}(\psi)(\xi)$$

so

4 $$\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\cos(c\xi t) + \frac{\mathcal{F}(\psi)(\xi)}{c\xi}\sin(c\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\sin(c\xi(t-\tau))\,d\tau.$$

(OVER)

Next we use the identities

16 $$\cos(c\xi t) = \frac{1}{2}e^{ic\xi t} + \frac{1}{2}e^{-ic\xi t} \quad \text{and} \quad \sin(c\xi t) = \frac{1}{2i}e^{ic\xi t} - \frac{1}{2i}e^{-ic\xi t}$$

and the Fourier transform formulas

18 $$\mathcal{F}\big(g(x-a)\big)(\xi) = \hat{g}(\xi)e^{-i\xi a} \quad \text{and} \quad \mathcal{F}\Big(\int_{-\infty}^{x} g(s)\,ds\Big)(\xi) = \frac{\hat{g}(\xi)}{i\xi}$$

(see the homework exercises on Fourier transforms) to write

$$\mathcal{F}(u)(\xi) = \frac{1}{2}\mathcal{F}(\varphi)(\xi)e^{ic\xi t} + \frac{1}{2}\mathcal{F}(\varphi)(\xi)e^{-ic\xi t} + \frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x}\psi(s)\,ds\Big)(\xi)e^{ic\xi t} \qquad e^{-ic\xi(t-\tau)}$$

+7 $$-\frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x}\psi(s)\,ds\Big)(\xi)e^{-ic\xi t} + \int_0^t\Big[\frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x} f(s,\tau)\,ds\Big)(\xi)e^{ic\xi(t-\tau)} - \frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x} f(s,\tau)\,ds\Big)(\xi)$$

$$= \frac{1}{2}\mathcal{F}\big(\varphi(x+ct)\big)(\xi) + \frac{1}{2}\mathcal{F}\big(\varphi(x-ct)\big)(\xi) + \frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x+ct}\psi(s)\,ds\Big)(\xi)$$

$$-\frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x-ct}\psi(s)\,ds\Big)(\xi) + \int_0^t\Big[\frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x+c(t-\tau)} f(s,\tau)\,ds\Big)(\xi) - \frac{1}{2c}\mathcal{F}\Big(\int_{-\infty}^{x-c(t-\tau)} f(s,\tau)\,ds\Big)(\xi)\Big]d\tau$$

22 $$= \mathcal{F}\Big(\frac{1}{2}\big[\varphi(x+ct)+\varphi(x-ct)\big]\Big)(\xi) + \mathcal{F}\Big(\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds\Big)(\xi)$$

$$+ \int_0^t \mathcal{F}\Big(\frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\Big)(\xi)\,d\tau\,.$$

Interchanging the order of integration in the last term and using linearity of the Fourier transform yields

+10 $$\mathcal{F}(u)(\xi) = \mathcal{F}\Big(\frac{1}{2}\big[\varphi(x+ct)+\varphi(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau\Big)(\xi)\,.$$

Applying the inversion formula leads to

5 $$\boxed{u(x,t) = \frac{1}{2}\big[\varphi(x+ct)+\varphi(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau\,.}$$

4.(25 pts.) Consider the partial differential equation $u_t - u_{xx} = 0$ in $0 < x < 1,\ 0 < t < 10$, subject to the boundary conditions $u_x($0,t [redacted]$) = 0 = u(1,t)$ if $0 \le t \le 10$ and the initial condition

$$u(x,0) = 3\cos\left(\frac{\pi x}{2}\right) - 2\cos\left(\frac{5\pi x}{2}\right) \quad \text{if } 0 \le x \le 1.$$

15 (a) Use the method of separation of variables and show ALL DETAILS of the calculation of the eigenvalues and eigenfunctions for this problem. (Note well that the boundary condition at the left endpoint is a Neumann condition and that at the right endpoint is a Dirichlet condition.)

10 (b) Write a formula for the solution $u = u(x,t)$ to this problem.

BONUS: For 10 extra points, show that there is at most one solution to this problem.

(a) Consider nontrivial solutions to the homogeneous portion of this problem of the form $u(x,t) = X(x)T(t)$. Substituting this in the PDE gives $X(x)T'(t) - X''(x)T(t) = 0$ $\Rightarrow -\frac{T'(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \text{constant} = \lambda$. Substituting the functional form for u in the boundary conditions produces $\left\{ \begin{array}{l} 0 = u_x(0,t) = X'(0)T(t) \\ 0 = u(1,t) = X(1)T(t) \end{array} \right\}$ for all $0 \le t \le 10$. Since u is nontrivial, it follows that $X'(0) = 0 = X(1)$. Therefore we have the coupled system

5 $$\left\{ \begin{array}{l} X''(x) + \lambda X(x) = 0, \quad X'(0) = 0,\ X(1) = 0, \\ T'(t) + \lambda T(t) = 0. \end{array} \right.$$

We assume that the eigenvalues are real.

<u>Case 1</u>: $\lambda = 0$. $X''(x) = 0 \Rightarrow X'(x) = c_1$ and $X(x) = c_1 x + c_2$. Then $0 = X'(0) = c_1$

7 and $0 = X(1) = \cancel{c_1}^{0}(1) + c_2 = c_2$. There are only trivial solutions in this case; i.e. $\lambda = 0$ is not an eigenvalue.

<u>Case 2</u>: $\lambda < 0$, say $\lambda = -\alpha^2$ where $\alpha > 0$. $X''(x) - \alpha^2 X(x) = 0 \Rightarrow X(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$ and $X'(x) = \alpha c_1 \sinh(\alpha x) + \alpha c_2 \cosh(\alpha x)$. Then $0 = X'(0) = \alpha c_2 \Rightarrow c_2 = 0$ and

9 $0 = X(1) = c_1\cosh(\alpha) + \cancel{c_2}^{0}\sinh(\alpha) = c_1 \overbrace{\cosh(\alpha)}^{\text{positive}} \Rightarrow c_1 = 0$. There are only trivial solutions in this case as well, so there are no negative eigenvalues.

<u>Case 3</u>: $\lambda > 0$, say $\lambda = \alpha^2$ where $\alpha > 0$. $X''(x) + \alpha^2 X(x) = 0 \Rightarrow X(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$ and $X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$. Then $0 = X'(0) = \alpha c_2 \Rightarrow c_2 = 0$ and $0 = X(1) = c_1\cos(\alpha) + \cancel{c_2}^{0}\sin(\alpha) = c_1\cos(\alpha)$. For nontrivial solutions, we must have $\cos(\alpha) = 0$ and hence $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ Therefore:

eigenvalues: $\lambda_n = \alpha_n^2 = \left(\frac{(2n-1)\pi}{2}\right)^2 \quad (n = 1,2,3,\ldots)$ (OVER)

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(b) The t-equation is $T_n'(t) + \lambda_n T_n(t) = 0 \Rightarrow T_n'(t) + \left(\frac{(2n-1)\pi}{2}\right)^2 T_n(t) = 0$

$\Rightarrow T_n(t) = e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$ (up to a constant factor) for $n = 1, 2, 3, \ldots$

4 Therefore $u_n(x,t) = X_n(x) T_n(t) = \cos\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$ $(n = 1, 2, 3, \ldots)$

solves the homogeneous portion of the problem. The superposition principle implies that

5 $$u(x,t) = \sum_{n=1}^{N} c_n \cos\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$$

solves the homogeneous part of the problem for each integer $N \geq 1$ and all constants $c_1, \ldots, c_N$. We want

6 $$3\cos\left(\frac{\pi x}{2}\right) - 2\cos\left(\frac{5\pi x}{2}\right) = u(x,0) = \sum_{n=1}^{N} c_N \cos\left(\frac{(2n-1)\pi x}{2}\right)$$

for all $0 \leq x \leq 1$. By inspection, we may take $N = 3$ and $c_1 = 3$, $c_2 = 0$, $c_3 = -2$. Consequently, a solution to the entire problem is

10 $$\boxed{u(x,t) = 3\cos\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2 t}{4}} - 2\cos\left(\frac{5\pi x}{2}\right) e^{-\frac{25\pi^2 t}{4}}}.$$

<u>BONUS</u>. Suppose $u = v(x,t)$ were another solution to the problem and consider

2 $w(x,t) = u(x,t) - v(x,t)$. Then w solves

4 $$\begin{cases} w_t - w_{xx} = 0 & \text{in } R: 0 < x < 1,\ 0 < t \leq 10, \\ w_x(0,t) = 0 = w(1,t) & \text{if } 0 \leq t \leq 10, \\ w(x,0) = 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

The energy of the solution w at time $t \geq 0$ is given by

5 $$E(t) = \int_0^1 w^2(x,t)\,dx,$$ (CONT.)

so $$\frac{dE}{dt} = \frac{d}{dt}\int_0^1 w^2(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}\Big(w^2(x,t)\Big)dx = \int_0^1 2w(x,t)w_t(x,t)\,dx =$$

$$\int_0^1 \underbrace{2w(x,t)}_{U}\,\underbrace{w_{xx}(x,t)\,dx}_{dV} = 2w(x,t)w_x(x,t)\Big|_{x=0}^{1} - 2\int_0^1 w_x^2(x,t)\,dx =$$

$$2\cancelto{0}{w(1,t)}w_x(1,t) - 2w(0,t)\cancelto{0}{w_x(0,t)} - 2\int_0^1 w_x^2(x,t)\,dx = -2\int_0^1 w_x^2(x,t)\,dx \le 0.$$

8 That is, the energy is a decreasing function of t in $[0,10]$. Therefore

$0 \le \int_0^1 w^2(x,t)\,dx = E(t) \le E(0) = \int_0^1 w^2(x,0)\,dx = \int_0^1 0\,dx = 0$ for all $0 \le t \le 10$,

so $\int_0^1 w^2(x,t)\,dx \equiv 0$. The vanishing theorem then implies $w(x,t)=0$

for all $0 \le x \le 1$ and each fixed $t \in [0,1]$. It follows that $v(x,t)=u(x,t)$

10 for all (x,t) in $\overline{R}$; i.e. the solution to the problem is unique.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$