Mathematics 325 | Exam III Fall 2005 | Name: Dr. Grow

1.(25 pts.) Solve the eigenvalue problem $X''(x)+\lambda X(x)=0$ for $0<x<\pi$, subject to the Neumann boundary condition $X'(0)=0$ at the left endpoint and the Dirichlet condition $X(\pi)=0$ at the right endpoint. (Note: If you want full credit, you will need to show the details of **all** steps in your solution.)

$\underline{\lambda>0}$ (say $\lambda=\alpha^2>0$ where $\alpha>0$):

$X''+\alpha^2 X=0 \Rightarrow X(x)=c_1\cos(\alpha x)+c_2\sin(\alpha x)$ so $X'(x)=-\alpha c_1\sin(\alpha x)+\alpha c_2\cos(\alpha x)$. (2 pts. to here.) (3 pts. to here.)

$0=X'(0)=\alpha c_2 \Rightarrow c_2=0$ so $X(x)=c_1\cos(\alpha x)$. $0=X(\pi)=c_1\cos(\alpha\pi)\Rightarrow \cos(\alpha\pi)=0$ (4 pts. to here.) (6 pts. to here.) (necessary for a nontrivial solution)

$\Rightarrow \alpha\pi=(2k+1)\frac{\pi}{2} \Rightarrow \alpha=\frac{2k+1}{2}$. 9 pts. to here.

15 pts.

$$\boxed{\begin{array}{l}\text{Eigenvalues: } \lambda_k=\left(\frac{2k+1}{2}\right)^2 \\ \text{Eigenfunctions: } X_k(x)=\cos\left(\left(\frac{2k+1}{2}\right)x\right)\end{array}\right\}\ k=0,1,2,\ldots}$$

(11 pts. to here.) (14 pts. to here) (15 pts. to here.)

5 pts.

$\underline{\lambda=0}$: $X''=0 \Rightarrow X(x)=c_1x+c_2$ so $X'(x)=c_1$ (2 pts. to here)

$0=X'(0)=c_1$ so $X(x)=c_2$. $0=X(\pi)=c_2$ (3 pts to here.) (4 pts. to here.)

$\therefore$ $\boxed{\text{zero is not an eigenvalue.}}$ 5 pts. to here.

5 pts.

$\underline{\lambda<0}$ (say $\lambda=-\alpha^2$ where $\alpha>0$):

$X''-\alpha^2X=0 \Rightarrow X(x)=c_1\cosh(\alpha x)+c_2\sinh(\alpha x)$ so $X'(x)=\alpha c_1\sinh(\alpha x)+\alpha c_2\cosh(\alpha x)$. (2 pts to here)

$0=X'(0)=\alpha c_2 \Rightarrow c_2=0$ so $X(x)=c_1\cosh(\alpha x)$. $0=X(\pi)=c_1\underbrace{\cosh(\alpha\pi)}_{\text{positive}} \Rightarrow c_1=0$. (3 pts. to here.) (4 pts to here.)

$\boxed{\therefore \text{ there are no negative eigenvalues.}}$

5 pts. to here.

IMPORTANT NOTE: If you recognize the eigenvalues and eigenfunctions for any boundary value problem on the remainder of this exam, you do **NOT** need to supply all the details of the calculations. Merely state the eigenvalues and eigenfunctions and proceed with the rest of the solution.

2.(25 pts.) Find a solution to $u_{tt} - u_{xx} \overset{①}{=} 0$ for $0 < x < \pi$, $0 < t < \infty$, subject to the boundary conditions $u_x(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(\pi,t)$ for $t \geq 0$, and the initial conditions $u(x,0) \overset{⑤}{=} 3 + 2\cos(x) - 5\cos(3x)$, $u_t(x,0) \overset{④}{=} 0$ for $0 \leq x \leq \pi$.

$u(x,t) = X(x)T(t)$ in ①-②-③-④ leads to $X(x)T''(t) - X''(x)T(t) = 0 \Rightarrow$

2 pts. to here

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda \quad \text{so} \quad \boxed{X''(x) + \lambda X(x) = 0, \quad X'(0) \overset{\text{by ②}}{=} 0 \text{ and } X'(\pi) \overset{\text{by ③}}{=} 0}$$

(2+1 6 to

$$T''(t) + \lambda T(t) = 0, \quad T'(0) \overset{\text{by ④}}{=} 0 .$$

9 pts. to here (2+1)

11 pts. to here. The eigenvalues of the boxed BVP are $\lambda_n = n^2$ $(n = 0, 1, 2, \ldots)$ and the corresponding

13 pts. to here. eigenfunctions are $X_n(x) = \cos(nx)$.

The solution of the corresponding equation in t, $T_n''(t) + n^2 T_n(t) = 0$, is $T_n(t) = c_1\cos(nt) + c_2\sin(nt)$ and the condition $T_n'(0) = 0$ implies $c_2 = 0$, at least when $n = 1, 2, 3, \ldots$ If $n = 0$ then

15 pts. to here. $T_0(t) = c_2 t + c_1$ and $T_0'(0) = 0$ implies $c_2 = 0$. Therefore $T_n(t) = \cos(nt)$ $(n = 0, 1, 2, \ldots)$,

17 pts to here up to a constant factor. Hence $u_n(x,t) = X_n(x)T_n(t) = \cos(nx)\cos(nt)$ $(n = 0, 1, 2, \ldots)$.

By the superposition principle, for each $N \geq 1$ and each choice of constants $c_0, c_1, \ldots, c_N$,

19 pts. to here.

$$(*) \qquad u(x,t) = \sum_{n=0}^{N} c_n \cos(nx)\cos(nt)$$

solves ①-②-③-④. We want to choose N and $c_0, \ldots, c_N$ such that $(*)$ satisfies the nonhomogeneous condition ⑤. I.e.

$$3 + 2\cos(x) - 5\cos(3x) = u(x,0) = \sum_{n=0}^{N} c_n \cos(nx) \quad \text{for all } 0 \leq x \leq \pi.$$

23 pts. to here By inspection we may take $N = 3$ and $c_0 = 3$, $c_1 = 2$, $c_2 = 0$, $c_3 = -5$. Therefore

25 pts. to here

$$\boxed{u(x,t) = 3 + 2\cos(x)\cos(t) - 5\cos(3x)\cos(3t)}$$

is a solution of ①-②-③-④-⑤.

3.(25 pts.) (a) Show that $\Phi=\{\sin(n\pi x)\}_{n=1}^{\infty}$ is an orthogonal set of functions on the interval $(0,1)$.

(Hint: You may find useful the identity $\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$.)

(b) Consider the function $f(x)=x(1-x)$ on $0\le x\le 1$. Show that the Fourier sine series of f on the interval $(0,1)$ is $f(x)\sim\sum_{k=0}^{\infty}\frac{8\sin((2k+1)\pi x)}{\pi^3(2k+1)^3}$.

(c) Assuming that $f(x)=x(1-x)$ is equal to its Fourier sine series at each point of the interval $0\le x\le 1$, show that $\frac{\pi^3}{32}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

5 pts (a) If $m\neq n$ then $\langle \varphi_m,\varphi_n\rangle=\int_0^1 \varphi_m(x)\overline{\varphi_n(x)}\,dx=\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx=$

$$\int_0^1 \frac{1}{2}\left[\cos((m-n)\pi x)-\cos((m+n)\pi x)\right]dx=\left[\frac{\sin((m-n)\pi x)}{2(m-n)\pi}-\frac{\sin((m+n)\pi x)}{2(m+n)\pi}\right]\Bigg|_0^1=0.$$

1 pt. to here. 2 pts. to here 4 pts to here.

10 pts. (b) $$b_n=\frac{\langle f,\varphi_n\rangle}{\langle\varphi_n,\varphi_n\rangle}=2\int_0^1 \overbrace{x(1-x)}^{u}\,\overbrace{\sin(n\pi x)dx}^{dV}=\cancelto{0}{\frac{-2x(1-x)\cos(n\pi x)}{n\pi}\Big|_0^1}+2\int_0^1\underbrace{(1-2x)}_{u}\underbrace{\frac{\cos(n\pi x)}{n\pi}dx}_{dV}$$

6 pts. to here 7 pts to here.

$$=\cancelto{0}{\frac{2(1-2x)}{n\pi}\cdot\frac{\sin(n\pi x)}{n\pi}\Big|_0^1}-\frac{2}{n\pi}\int_0^1\frac{\sin(n\pi x)}{n\pi}(-2dx)=\frac{4}{(n\pi)^2}\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1=\frac{4(1-(-1)^n)}{(n\pi)^3}$$

8 pts. to here.

$$\therefore\ b_n=\begin{cases}0 & \text{if } n=2k \text{ is even,}\\ \dfrac{8}{[(2k+1)\pi]^3} & \text{if } n=2k+1 \text{ is odd.}\end{cases}$$

Thus, the Fourier sine series $\sum_{n=1}^{\infty}b_n\sin(n\pi x)$ of f is $f(x)\sim\boxed{\sum_{k=0}^{\infty}\frac{8\sin((2k+1)\pi x)}{\pi^3(2k+1)^3}}$. 10 pts. to here

10 pts. (c) Assume $x(1-x)=f(x)=\sum_{k=0}^{\infty}\frac{8\sin((2k+1)\pi x)}{\pi^3(2k+1)^3}$ for all $0\le x\le 1$. Taking $x=1/2$ in this identity gives

5 pts. to here.

$$\frac{1}{4}=\sum_{k=0}^{\infty}\frac{8\sin((2k+1)\pi/2)}{\pi^3(2k+1)^3}\ \Rightarrow\ \boxed{\frac{\pi^3}{32}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}}.$$

10 pts. to here

4.(25 pts.) Find a solution to $u_t - 2u_{xx} \overset{1}{=} 0$ for $0 < x < 1$, $0 < t < \infty$, subject to the boundary conditions $u(0,t) \overset{2}{=} 0 \overset{3}{=} u(1,t)$ for $t \geq 0$, and the initial condition $u(x,0) \overset{4}{=} x(1-x)$ for $0 \leq x \leq 1$. (Note: You may find useful the result of problem # 3(b).)

We seek nontrivial solutions of 1-2-3 of the form $u(x,t) = X(x)T(t)$. Then 1 implies $X(x)T'(t) - 2X''(x)T(t) = 0 \Rightarrow \dfrac{T'(t)}{2T(t)} = \dfrac{X''(x)}{X(x)} = \text{constant} = -\lambda$. 2 implies $X(0)T(t) = 0$ and 3 implies $X(1)T(t) = 0$ for all $t \geq 0$. Therefore $X''(x) - \lambda X(x) = 0$, $X(0) = 0 = X(1)$. 6 pts. to here

9 pts. to here

The eigenvalues for this B.V.P. are $\lambda_n = (n\pi)^2$ for $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $X_n(x) = \sin(n\pi x)$.

12 pts. to here

The t-equation, $T_n'(t) + 2\lambda_n T_n(t) = 0$, has solution $T_n(t) = e^{-2\lambda_n t} = e^{-2n^2\pi^2 t}$ (up to a constant factor).

15 pts. to here

Therefore, the superposition principle yields the formal solution $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-2n^2\pi^2 t}$ to the homogeneous problem 1-2-3 where $b_1, b_2, b_3, \ldots$ are arbitrary constants.

18 pts. to here

Applying 4 we have

$$x(1-x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for all } 0 \leq x \leq 1.$$

20 pts. to here

By #3(b) and assuming that $x(1-x)$ is equal to its Fourier sine series at each point of the interval $[0,1]$, $b_n = 0$ if n is even and $b_n = \dfrac{8}{\pi^3 n^3}$ if n is odd. That is,

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8\sin((2k+1)\pi x)e^{-2(2k+1)^2\pi^2 t}}{\pi^3(2k+1)^3}}$$

25 pts. to here

Bonus (25pts.): (a) Show that the solution to the problem in # 2 is unique.
(b) Show that the solution to the problem in # 4 is unique.

(a) Let $v = v(x,t)$ be another solution to the problem in #2, and let

2 pts. to here. $w(x,t) = u(x,t) - v(x,t)$ where $u(x,t) = 3 + 2\cos(x)\cos(t) - 5\cos(3x)\cos(3t)$ is the solution obtained in #2. Then $w = w(x,t)$ solves $w_{tt} - w_{xx} = 0$ in $0 < x < \pi$, $0 < t < \infty$,

5 and satisfies $w_x(0,t) = 0 = w_x(\pi,t)$ for $t \geq 0$ and $w(x,0) = 0 = w_t(x,0)$ for $0 \leq x \leq \pi$.

8 Consider the energy function $E(t) = \int_0^\pi \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx$ of the solution $w = w(x,t)$. Then

$$\frac{dE}{dt} = \int_0^\pi \left[w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{xt}(x,t)\right]dx$$

$$= \int_0^\pi \left[w_t(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t)\right]dx = \int_0^\pi \frac{\partial}{\partial x}(w_t w_x)\,dx = w_t(x,t)w_x(x,t)\Big|_{x=0}^{\pi}$$

10 $= w_t(\pi,t)\cancelto{0}{w_x(\pi,t)} - w_t(0,t)\cancelto{0}{w_x(0,t)} = 0$ so $E(t) = E(0)$ for all $t \geq 0$.

But $E(0) = \int_0^\pi \left[\frac{1}{2}\cancelto{0}{w_t^2(x,0)} + \frac{1}{2}\cancelto{0}{w_x^2(x,0)}\right]dx = 0$ so the vanishing theorem implies

11 $0 = w_t^2(x,t) = w_x^2(x,t)$ for all $0 \leq x \leq \pi$ and each $t \geq 0$. Hence $w(x,t) =$ constant in $0 \leq x \leq \pi$, $0 \leq t < \infty$. But $w(x,0) = 0$ for $0 \leq x \leq \pi$ so $w(x,t) = 0$ for all

12 $0 \leq x \leq \pi$, $0 \leq t < \infty$. That is, $u(x,t) = v(x,t)$ for $0 \leq x \leq \pi$, $0 \leq t < \infty$ so the solution is unique to the problem in #2.

13 pts to here.

2 (b) Let $w(x,t) = u(x,t) - v(x,t)$ where $u = u(x,t) = \sum_{k=0}^{\infty} \frac{8\sin((2k+1)\pi x)e^{-2(2k+1)^2\pi^2 t}}{\pi^3(2k+1)^3}$ is the solution obtained in problem #4 and $v = v(x,t)$ is any other solution to the problem in #4. Then $w = w(x,t)$ solves $w_t - 2w_{xx} = 0$ in $0 < x < 1$, $0 < t < \infty$,

6 and satisfies $w(0,t) = 0 = w(1,t)$ for $t \geq 0$, and $w(x,0) = 0$ for $0 \leq x \leq 1$. (OVER)

Since $w = 0$ on the side walls $x = 0$ and $x = 1$ and $w = 0$ on the initial wall $t = 0$, the maximum/minimum principle for solutions of the diffusion equation shows that

11 $w(x,t) = 0$ for all $0 \le x \le 1$, $0 \le t \le T$, where $T > 0$ is arbitrary. It follows that $u(x,t) = v(x,t)$ for all $0 \le x \le 1$ and $0 \le t < \infty$;

12 pts. to here. i.e. the solution to the problem in #4 is unique.

Math 225

Exam III

Fall 2005

Mean: 76.2

Standard Deviation: 26.4

(n = 18)

Distribution of Scores:

	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87 or above	A	A	9
73 – 86	B	B	1
60 – 72	C	B	2
50 – 59	C	C	2
0 – 49	F	D	4