Mathematics 325 Exam III Name: Dr. Grow
Summer 2002 (1 pt.)

3.(26 pts.) In the following problem, you may USE the following facts WITHOUT PROOF. The operator $Tf = -f''$ is hermitian on

$$V = \{ f \in C^2[0,1] : f(0) = 0 = f'(1) \},$$

equipped with the standard inner product

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}dx .$$

The eigenvalues of T on V are $\lambda_n = (n + 1/2)^2\pi^2$ and the eigenfunctions are $X_n(x) = \sin((n + 1/2)\pi x)$ $(n = 0,1,2,...)$.

(a) Why is $\{ X_n \}_{n=1}^{\infty}$ an orthogonal set of functions on $[0,1]$?

(b) Show that the generalized Fourier series representation of $\phi(x) = x(2-x)$ on $[0,1]$ with respect to $\{ X_n \}_{n=1}^{\infty}$ is

$$\phi(x) \sim \sum_{n=0}^{\infty} \frac{32\sin((n + 1/2)\pi x)}{(2n + 1)^3\pi^3} .$$

(c) Write the sum of the first three terms of the above generalized Fourier series for ϕ. Sketch the graph of this sum on $[0,1]$, and on the same coordinate axes, sketch the graph of ϕ.

(d) Does the generalized Fourier series of ϕ converge to ϕ uniformly on $[0,1]$? Why?

(e) Does the generalized Fourier series of ϕ converge to ϕ in the mean square sense on $[0,1]$? Why?

(f) Does the generalized Fourier series of ϕ converge to ϕ pointwise on $[0,1]$? Why?

(g) Use the results above to help evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

(h) Use the results above to help evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}$.

(a) It is a theorem that eigenfunctions of a hermitian operator that correspond to distinct eigenvalues are orthogonal. Hence $\int_0^1 X_n(x)\overline{X_m(x)}dx = \langle X_n, X_m\rangle = 0$ if $m \neq n$.

(b) $$A_n = \frac{\langle \varphi, X_n\rangle}{\langle X_n, X_n\rangle} = 2\int_0^1 \overbrace{x(2-x)}^{u}\,\overbrace{\sin((n+\tfrac{1}{2})\pi x)\,dx}^{dv} = \cancel{\left.\frac{-2x(2-x)\cos((n+\frac12)\pi x)}{(n+\frac12)\pi}\right|_0^1} + 2\int_0^1 \overbrace{(2-2x)}^{u}\,\overbrace{\frac{\cos(\ldots)}{(n+\frac12)\pi}}^{dv}$$

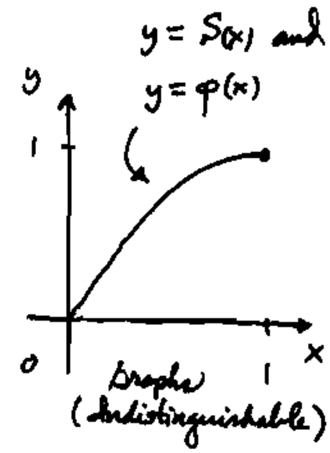
(cont.)

$$\langle X_n, X_n\rangle = \int_0^1 \sin^2((n+\tfrac12)\pi x)dx = \int_0^1\left[\frac12 - \frac12\cos(2(n+\tfrac12)\pi x)\right]dx = \left.\frac{x}{2} - \frac{\sin((2n+1)\pi x)}{2(2n+1)\pi}\right|_0^1 = \frac12$$

$$A_n = \cancel{\frac{2(2-2x)\sin((n+\frac12)\pi x)}{(n+\frac12)^2\pi^2}}\Bigg|_0^{1} \xrightarrow{} 0 + 2\int_0^1 \frac{\sin((n+\frac12)\pi x)}{(n+\frac12)^2\pi^2}(+2\,dx) = \frac{-4\cos((n+\frac12)\pi x)}{(n+\frac12)^3\pi^3}\Bigg|_0^1 = \frac{4}{(n+\frac12)^3\pi^3}$$

$$A_n = \frac{8\cdot 4}{8\cdot(n+\frac12)^3\pi^3} = \boxed{\frac{32}{(2n+1)^3\pi^3}} \qquad (n=0,1,2,\dots)$$

$$\therefore \quad \boxed{\varphi(x) \sim \sum_{n=0}^{\infty} \frac{32\sin((n+\frac12)\pi x)}{(2n+1)^3\pi^3}}$$



3 (c) $\boxed{S(x) = \frac{32}{\pi^3}\left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{27}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{125}\sin\left(\frac{5\pi x}{2}\right)\right)}$

3 (d) (i) $\varphi(x) = x(2-x)$, $\varphi'(x) = 2-2x$, $\varphi''(x) = -2$ exist and are continuous on $[0,1]$.

(ii) $\varphi(0) = 0$, $\varphi'(1) = 0$ so φ satisfies the boundary conditions of ∇.

By Theorem 2, the generalized Fourier series of φ in part (b) converges uniformly to φ on $[0,1]$

3 (e) Yes, the generalized Fourier series of φ converges in the mean square sense to φ on $[0,1]$ because uniform convergence (see part (d)) implies mean square convergence (on bounded intervals)

3 (f) Yes, the generalized Fourier series of φ converges pointwise to φ on $[0,1]$ because uniform convergence (see part (d)) implies pointwise convergence.

3 (g) $x(2-x) = \sum_{n=0}^{\infty} \frac{32\sin((n+\frac12)\pi x)}{(2n+1)^3\pi^3}$ for all $0 \le x \le 1$ by part (f). Taking $x=1$ gives the desired result: $1 = \sum_{n=0}^{\infty} \frac{32\overbrace{\sin((n+\frac12)\pi)}^{(-1)^n}}{(2n+1)^3\pi^3} \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \boxed{\frac{\pi^3}{32}}$.

4 (h) By Parseval's identity, $\sum_{n=0}^{\infty}\left|\frac{32}{(2n+1)^3\pi^3}\right|^2 \overbrace{\int_0^1 \sin^2((n+\tfrac12)\pi x)\,dx}^{1/2} = \int_0^1 [x(2-x)]^2\,dx$.

$$\therefore \sum_{n=0}^{\infty}\frac{512}{(2n+1)^6\pi^6} = \frac{8}{15} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{8\pi^6}{15\cdot 512} = \boxed{\frac{\pi^6}{960}}$$

4.(26 pts.) Find a solution to $u_{tt} - u_{xx} = 0$ for $0 < x < 1$, $0 < t < \infty$, which satisfies $u(0,t) = 0$, $u_x(1,t) = 0$ for $t \geq 0$, and $u(x,0) = x(2-x)$, $u_t(x,0) = 0$ for $0 \leq x \leq 1$. (Hint: You may find some of the results of problem 3 useful.)

Bonus (10 pts.): How many solutions are there to the problem in part (a)? Give reasons for your answer.

We seek nontrivial solutions to the homogeneous portion of the problem of the form $u(x,t) = X(x)T(t)$. Separating variables in the PDE and applying the homogeneous boundary/initial conditions gives

4 $$\begin{cases} X''(x) + \lambda X(x) = 0, & X(0) = 0 = X'(1), \\ T''(t) + \lambda T(t) = 0, & T'(0) = 0. \end{cases}$$

8 By problem 3, the eigenvalues and eigenfunctions are $\lambda_n = (n+\frac{1}{2})^2\pi^2$ and $X_n(x) = \sin((n+\frac{1}{2})\pi x)$,

11 $n = 0,1,2,\dots$. The solution to the t-equation is $T_n(t) = \cos((n+\frac{1}{2})\pi t)$ $(n = 0,1,2,\dots)$. Thus

16 $$u(x,t) = \sum_{n=0}^{\infty} A_n \sin((n+\tfrac{1}{2})\pi x)\cos((n+\tfrac{1}{2})\pi t)$$

is the formal solution to the homogeneous portion of the problem. We need to choose the coefficients so the nonhomogeneous initial condition is satisfied:

19 $$x(2-x) = u(x,0) = \sum_{n=0}^{\infty} A_n \sin((n+\tfrac{1}{2})\pi x) \quad \text{for } 0 \leq x \leq 1.$$

22 By problem 3, $A_n = \dfrac{32}{(2n+1)^3\pi^3}$ for $n = 0,1,2,\dots$ Hence

26 $$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{32 \sin((n+\frac{1}{2})\pi x)\cos((n+\frac{1}{2})\pi t)}{(2n+1)^3\pi^3}}.$$

Bonus: The energy method shows that there is only one solution to the problem in part (a). (See me for details.)

5.(26 pts.) The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Centigrade on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies $u_r = -\gamma$ where γ is a positive constant.

18 (a) Find the temperature distribution function for the material.

\+ (b) What are the hottest and coldest temperatures in the material?

4 (c) Is it possible to choose γ so that the temperature on the outer boundary is 20 degrees Centigrade? Support your answer.

(a) The temperature distribution function for the material satisfies

$$\begin{cases} \nabla^2 u = 0 & \text{for } 1 < r < 2,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi, \\ u(1;\theta,\varphi) = 100 & \text{for } 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi, \\ u_r(2;\theta,\varphi) = -\gamma & \text{for } 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi. \end{cases}$$

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10 We assume that the solution is radial, i.e. $u = u(r)$, independent of the angles. Therefore

$$0 = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right)}^{\,0} + \cancel{\frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}}^{\,0}$$

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$$\Rightarrow \quad r^2 \frac{\partial u}{\partial r} = c_1 \quad \Rightarrow \quad u = \int \frac{c_1}{r^2}\, dr = \frac{-c_1}{r} + c_2$$

$$-\gamma = u_r(2) = \frac{c_1}{2^2} \quad \Rightarrow \quad c_1 = -4\gamma$$

$$100 = u(1) = +\frac{4\gamma}{1} + c_2 \quad \Rightarrow \quad c_2 = 100 - 4\gamma$$

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$$\therefore \quad \boxed{u(r) = \frac{4\gamma}{r} + 100 - 4\gamma}$$

2 (b) Hottest temperature: $u(1) = \frac{4\gamma}{1} + 100 - 4\gamma = \boxed{100}\ ^\circ C$

2 Coldest temperature: $u(2) = \frac{4\gamma}{2} + 100 - 4\gamma = \boxed{100 - 2\gamma}\ ^\circ C$

4 (c) Yes. $20 = u(2) = 100 - 2\gamma \Rightarrow \boxed{\gamma = 40}$

Check: $u(r) = \frac{160}{r} - 60$

$$u(2) = \frac{160}{2} - 60 \overset{\checkmark}{=} 20.$$

Mathematics 325 Take-Home Portion of Exam III Summer 2002 Name: Dr. Grow (1 pt.)

1.(10 pts.) (a) Let n be a nonnegative integer. Show that the operator
3 T given by

$$Tf(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r) \qquad (0 < r \le 1)$$

is hermitian on the vector space

$$V_B = \{\, f \in C^2(0,1] : f(1) = 0,\ f \text{ and } f' \text{ bounded on } (0,1] \,\}$$

equipped with the inner product

$$(*) \qquad \langle f,g\rangle = \int_0^1 f(r)\overline{g(r)}\,r\,dr.$$

2 (b) Are the eigenvalues of T on V_B real numbers?

3 (c) Are the eigenvalues of T on V_B positive?

2 (d) Are the eigenfunctions of T on V_B, corresponding to distinct eigenvalues, orthogonal on (0,1) relative to the inner product (*)?

(Please give reasons for your answers to (b)-(d).)

2.(10 pts.) Use separation of variables to solve the variable density vibrating string problem:

$$\frac{1}{(1+x)^2}u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0 = u(1,t) \qquad \text{for } 0 \le t < \infty,$$

$$u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \qquad \text{for } 0 \le x \le 1.$$

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$$u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \qquad \text{for } 0 \le x \le 1.$$

1. (a) Let f and g belong to V_B. Then

$$\langle Tf, g\rangle = \int_0^1 \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right]\overline{g(r)}\, r\, dr$$

$$= \int_0^1 \underbrace{\overline{g(r)}}_{v}\, \underbrace{\frac{d}{dr}\left(r\frac{df}{dr}\right)dr}_{dV} - \int_0^1 \frac{n^2}{r^2} f(r)\overline{g(r)}\, r\, dr$$

$$= \overline{g(r)}\, r\frac{df}{dr}\Big|_0^1 - \int_0^1 \overline{\frac{dg}{dr}}\, r\frac{df}{dr}\, dr - \int_0^1 \frac{n^2}{r^2} f(r)\overline{g(r)}\, r\, dr$$

Since $g(1) = 0$ and since $\overline{g(r)}f'(r)$ is bounded on $(0,1]$ we have $\lim_{r\to 0^+} \overline{g(r)}\, r\, f'(r) = 0$, and hence $\overline{g(r)}\, r\frac{df}{dr}\Big|_0^1 = 0$.

$$\therefore \langle Tf, g\rangle = -\int_0^1 \underbrace{\overline{\frac{dg}{dr}}}_{v}\, r\, \underbrace{\frac{df}{dr}\, dr}_{dV} - \int_0^1 \frac{n^2}{r^2} f(r)\overline{g(r)}\, r\, dr$$

$$= -\overline{g'(r)}\, r\, f(r)\Big|_0^1 + \int_0^1 f(r)\frac{d}{dr}\left(r\overline{\frac{dg}{dr}}\right)dr - \int_0^1 \frac{n^2}{r^2} f(r)\overline{g(r)}\, r$$

Since $f(1) = 0$ and since $\overline{g'(r)}f(r)$ is bounded on $(0,1]$ we have $\lim_{r\to 0^+} -\overline{g'(r)}\, r\, f(r) = 0$, and thus $-\overline{g'(r)}\, r\, f(r)\Big|_0^1 = 0$.

$$\therefore \langle Tf, g\rangle = \int_0^1 f(r)\left[\frac{1}{r}\frac{d}{dr}\left(r\overline{\frac{dg}{dr}}\right) - \frac{n^2}{r^2}\overline{g(r)}\right] r\, dr = \langle f, Tg\rangle.$$

Consequently, T is hermitian on V_B.

(b) Yes, by the argument given in class to prove Theorem 2, Sec 5.3, the eigenvalues of T on V_B are real numbers.

[**Theorem 2**: Let A be a hermitian operator on a vector subspace V of $C[a,b]$. Then all the eigenvalues of A are real numbers.

Proof: Let λ be an eigenvalue of A on V. Then there exists a nonzero function f in V such that $Af = \lambda f$. Therefore

$$\lambda\langle f,f\rangle = \langle \lambda f, f\rangle = \langle Af, f\rangle = \langle f, Af\rangle = \langle f, \lambda f\rangle = \bar{\lambda}\langle f,f\rangle.$$

Since $\langle f,f\rangle > 0$, it follows that $\lambda = \bar{\lambda}$; i.e. λ is a real number.]

(c) No, the eigenvalues of T on V_B are all negative, by an argument analogous to that given in class to prove Theorem 3, Sec. 5.3.

Reason: Let λ be an eigenvalue of T on V_B; from (b) we know that λ is a real number. We claim first that there is a <u>real-valued</u> eigenfunction φ for T on V_B corresponding to λ. Let f be a nonzero function in V_B such that $Tf = \lambda f$. Note that $T\bar{f} = \overline{Tf} = \overline{\lambda f} = \lambda \bar{f}$ and $\bar{f}$ belongs to V_B. Therefore $\bar{f}$ is also an eigenfunction of T on V_B corresponding to λ. Thus, at least one of the functions

$$\varphi = \mathrm{Re}(f) = \tfrac{1}{2}(f+\bar{f}) \quad \text{and} \quad \varphi = \mathrm{Im}(f) = \tfrac{1}{2i}(f-\bar{f})$$

is a nonzero real-valued function in V_B such that $T\varphi = \lambda\varphi$. This establishes the claim.

Next, observe that

$$\begin{aligned}
\lambda\langle\varphi,\varphi\rangle &= \langle\lambda\varphi,\varphi\rangle \\
&= \langle T\varphi,\varphi\rangle \\
&= \int_0^1 \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) - \frac{n^2}{r^2}\varphi(r)\right]\varphi(r)\,r\,dr \\
&= \int_0^1 \underbrace{\varphi(r)}_{v}\underbrace{\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)dr}_{dV} - n^2\int_0^1 \frac{1}{r}\varphi^2(r)\,dr \\
&= r\varphi'(r)\varphi(r)\Big|_0^1 - \int_0^1 r\left(\varphi'(r)\right)^2 dr - n^2\int_0^1 \frac{1}{r}\varphi^2(r)\,dr
\end{aligned}$$

By the argument in part (a), $r\varphi'(r)\varphi(r)\Big|_0^1 = 0$. Thus

$$\lambda\langle\varphi,\varphi\rangle = -\int_0^1 r\left(\varphi'(r)\right)^2 dr - n^2\int_0^1 \frac{1}{r}\varphi^2(r)\,dr < 0.$$

Because $\langle\varphi,\varphi\rangle > 0$, it follows that $\lambda < 0$.

(d) Yes, by the argument given in class to prove Theorem 1 of Sec. 5.3, the eigenfunctions of T on V_B corresponding to distinct eigenvalues are orthogonal on $(0,1)$ relative to the inner product (*).

Theorem 1: Let A be a hermitian operator on a vector subspace V of $C[a,b]$ and let λ_1 and λ_2 be distinct eigenvalues of A. Then any eigenfunctions φ_1 and φ_2 of A, corresponding to λ_1 and λ_2, respectively, are orthogonal.

Proof: Note that $\lambda_1 \neq \lambda_2$, $A\varphi_1 = \lambda_1\varphi_1$, and $A\varphi_2 = \lambda\varphi_2$. Thus

$$\begin{aligned} \lambda_1\langle\varphi_1,\varphi_2\rangle = \langle\lambda_1\varphi_1,\varphi_2\rangle = \langle A\varphi_1,\varphi_2\rangle &= \langle\varphi_1,A\varphi_2\rangle \\ &= \langle\varphi_1,\lambda_2\varphi_2\rangle \\ &= \overline{\lambda_2}\langle\varphi_1,\varphi_2\rangle \\ &= \lambda_2\langle\varphi_1,\varphi_2\rangle. \end{aligned}$$

That is, $(\lambda_1-\lambda_2)\langle\varphi_1,\varphi_2\rangle = \lambda_1\langle\varphi_1,\varphi_2\rangle - \lambda_2\langle\varphi_1,\varphi_2\rangle = 0$.

Since $\lambda_1 - \lambda_2 \neq 0$, it follows that $\langle\varphi_1,\varphi_2\rangle = 0$; i.e. φ_1 and φ_2 are orthogonal.

2. $$\begin{cases} \dfrac{1}{(1+x)^2} u_{tt} - u_{xx} \overset{①}{=} 0 & \text{for } 0<x<1,\ 0<t<\infty, \\ u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t) & \text{for } 0 \le t < \infty, \\ u(x,0) \overset{④}{=} x(1-x)\sqrt{1+x} \text{ and } u_t(x,0) \overset{⑤}{=} 0 & \text{for } 0 \le x \le 1. \end{cases}$$

Let $u(x,t) = \mathrm{X}(x)T(t)$ be a nontrivial solution to the homogeneous part of the problem; i.e. to ①-②-③-⑤. Substituting this functional form for u into ① and applying ②-③-⑤ yields

$$\begin{cases} 0 \overset{⑥}{=} T''(t) + \lambda T(t), \quad T'(0) \overset{⑦}{=} 0 \\ \boxed{0 \overset{⑧}{=} (1+x)^2 \mathrm{X}''(x) + \lambda \mathrm{X}(x), \quad \mathrm{X}(0) \overset{⑨}{=} 0 \overset{⑩}{=} \mathrm{X}(1)} \quad \text{Eigenvalue Problem} \end{cases}$$

It is easy to check that the operator $T = -(1+x)^2 \dfrac{d^2}{dx^2}$ is hermitian on the vector space $V = \{ \varphi \in C^2[0,1] : \varphi(0) = 0 = \varphi(1) \}$ equipped with the inner product

$$(*) \qquad \langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}(1+x)^{-2}\,dx.$$

[Let f and g belong to V. Then two integrations by parts shows]

$$\langle Tf, g \rangle = \int_0^1 -(1+x)^2 f''(x)\overline{g(x)}(1+x)^{-2}\,dx$$

$$= \int_0^1 -f''(x)\overline{g(x)}\,dx$$

$$= \left[f(x)\overline{g'(x)} - f'(x)\overline{g(x)} \right] \Bigg|_0^1 - \int_0^1 f(x)\overline{g''(x)}\,dx .$$

Since $f(1) = g(1) = 0$ and $f(0) = g(0) = 0$, it follows that

$$\langle Tf, g\rangle = -\int_0^1 f(x)\overline{g''(x)}\,dx = \int_0^1 f(x)\left[\overline{-(1+x)^2 g''(x)}\right](1+x)^{-2}dx$$

$$= \langle f, Tg\rangle .$$

I.e. T is hermitian on V equipped with inner product $(*)$.

Therefore all the eigenvalues λ for the problem ⑧-⑨-⑩ are real.

We seek solutions to ⑧ of the form $X(x) = (1+x)^{\alpha}$ where α is a constant to be determined. Then $X'(x) = \alpha(1+x)^{\alpha-1}$ and $X''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, so substituting in ⑧ and simplifying yields

$$(1+x)^2\alpha(\alpha-1)(1+x)^{\alpha-2} + \lambda(1+x)^{\alpha} = 0$$

$$\left[\alpha(\alpha-1) + \lambda\right]\cancel{(1+x)^{\alpha}} = 0$$

$$\alpha^2 - \alpha + \lambda = 0$$

Roots: $\alpha_+ = \dfrac{1+\sqrt{1-4\lambda}}{2}$, $\alpha_- = \dfrac{1-\sqrt{1-4\lambda}}{2}$.

Case 1: The roots are real and distinct (i.e. $\lambda < 1/4$).

Then $\alpha_+ > \alpha_-$ so the general solution to ⑧ is

$$X(x) = c_1(1+x)^{\alpha_+} + c_2(1+x)^{\alpha_-}.$$

Applying ⑨ and ⑩ we have

$$\begin{cases} 0 = X(0) = c_1 + c_2 \\ 0 = X(1) = c_1 2^{\alpha_+} + c_2 2^{\alpha_-}. \end{cases}$$

Since $\begin{vmatrix} 1 & 1 \\ 2^{\alpha_+} & 2^{\alpha_-} \end{vmatrix} = 2^{\alpha_-} - 2^{\alpha_+} < 0$, it follows that the only solution to the system above is $c_1 = c_2 = 0$. That is, there are no eigenvalues $\lambda < 1/4$.

Case 2: There is a single real root of multiplicity two (i.e. $\lambda = 1/4$).

Then $\alpha_+ = \alpha_- = \frac{1}{2}$. The general solution to ⑧ is

$$X(x) = c_1(1+x)^{1/2} + c_2(1+x)^{1/2}\ln(1+x).$$

(The second solution $(1+x)^{1/2}\ln(1+x)$ can be obtained from the first $(1+x)^{1/2}$ by using the method of reduction of order.)

Applying ⑨ and ⑩ we have

$$\begin{cases} 0 = X(0) = c_1 \\ 0 = X(1) = c_1 2^{1/2} + c_2 2^{1/2}\ln(2) \end{cases} \Rightarrow c_1 = c_2 = 0$$

That is, $\lambda = 1/4$ is not an eigenvalue.

__Case 3__: The roots are complex and conjugate to one another (i.e. $\lambda > 1/4$).

Then $\alpha_{\pm} = \dfrac{1 \pm \sqrt{1-4\lambda}}{2} = \dfrac{1 \pm i\sqrt{4\lambda - 1}}{2} \equiv \dfrac{1 \pm i\mu}{2}$.

(Note that $\mu > 0$ and $\mu^2 = 4\lambda - 1$.)

The general solution to ⑧ is

$$X(x) = c_1(1+x)^{\frac{1+i\mu}{2}} + c_2(1+x)^{\frac{1-i\mu}{2}} = (1+x)^{1/2}\left[c_1(1+x)^{i\mu/2} + c_2(1+x)^{-i\mu/2}\right]$$

Note that $(1+x)^{i\mu/2} = \left[e^{\ln(1+x)}\right]^{i\mu/2} = e^{\frac{i\mu\ln(1+x)}{2}}$

$$= \cos\left(\frac{\mu\ln(1+x)}{2}\right) + i\sin\left(\frac{\mu\ln(1+x)}{2}\right)$$

and similarly $(1+x)^{-i\mu/2} = \cos\left(\dfrac{\mu\ln(1+x)}{2}\right) - i\sin\left(\dfrac{\mu\ln(1+x)}{2}\right)$.

Thus the general solution to ⑧ can be expressed as

$$X(x) = (1+x)^{1/2}\left[c_3\cos\left(\frac{\mu\ln(1+x)}{2}\right) + c_4\sin\left(\frac{\mu\ln(1+x)}{2}\right)\right]$$

Applying ⑨ and ⑩ yields

$$\begin{cases} 0 = X(0) = c_3 \\ 0 = X(1) = 2^{1/2}\left[c_3\cos\left(\dfrac{\mu\ln(2)}{2}\right) + c_4\sin\left(\dfrac{\mu\ln(2)}{2}\right)\right]. \end{cases}$$

Hence $c_3 = 0$ and $2^{1/2}c_4\sin\left(\dfrac{\mu\ln(2)}{2}\right) = 0$. To have a nontrivial solution we must have $\mu = \mu_n = \dfrac{2n\pi}{\ln(2)}$ $(n = 1, 2, 3, \ldots)$ Thus, the eigenvalues/eigenfunctions of ⑧-⑨-⑩ are

18 $(n = 1, 2, 3, \ldots)$ $\quad \lambda_n = \dfrac{\ln^2(2) + 4n^2\pi^2}{\cdots}$, $\quad X_n(x) = (1+x)^{1/2}\sin\left(n\pi\ln(1+x)\right)$

Returning to ⑥-⑦ with $\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{\ln^2(2)}$ $(n = 1, 2, 3, \ldots)$,

we readily see that, up to a constant factor,

$$T_n(t) = \cos\left(\sqrt{\frac{1}{4} + \frac{n^2\pi^2}{\ln^2(2)}}\; t\right).$$

Thus, a formal solution to ①-②-③-⑤ is

$$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t)$$

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$$= \sum_{n=1}^{\infty} b_n (1+x)^{1/2} \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right) \cos\left(\sqrt{\frac{1}{4} + \frac{n^2\pi^2}{\ln^2(2)}}\; t\right)$$

where $b_1, b_2, b_3, \ldots$ are "arbitrary" constants. We want to choose these coefficients so the nonhomogeneous initial condition ④ is met:

$$x(1-x)(1+x)^{1/2} = u(x,0) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0) = (1+x)^{1/2} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)$$

for all $0 \le x \le 1$. (I.e.

$$x(1-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right) \qquad (0 \le x \le 1).\ )$$

Note that the eigenfunctions $X_n(x) = (1+x)^{1/2} \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)$,

being eigenfunctions of the hermitian operator $T = -(1+x)^2 \frac{d^2}{dx^2}$

on V corresponding to distinct eigenvalues $\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{\ln^2(2)}$,

are orthogonal with respect to the inner product (*). Therefore, we should take

$$b_n = \frac{\left\langle x(1-x)(1+x)^{1/2},\ \Phi_n(x) \right\rangle}{\left\langle \Phi_n(x),\ \Phi_n(x) \right\rangle}$$

$$= \frac{\int_0^1 x(1-x)(1+x)^{1/2}(1+x)^{1/2} \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)(1+x)^{-2}\,dx}{\int_0^1 \left[(1+x)^{1/2} \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)\right]^2 (1+x)^{-2}\,dx}$$

$$= \frac{\int_0^1 x(1-x) \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)(1+x)^{-1}\,dx}{\int_0^1 \sin^2\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)(1+x)^{-1}\,dx} \qquad (n = 1, 2, 3, \dots)$$

$$1-4\lambda_n = -\mu_n^2$$

$$T_n''(t) + \lambda_n T_n(t) = 0$$

$$\Rightarrow \frac{1+\mu_n^2}{4} = \lambda_n$$

$$T_n''(t) + \left(\frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4}\right)T_n(t) = 0$$

$$\Rightarrow \frac{1 + \frac{4n^2\pi^2}{\ln^2(2)}}{4} = \lambda_n$$

$$T_n(t) = \alpha_n \cos\left(\sqrt{\frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4}}\, t\right) + \beta_n \sin\left(\sqrt{\frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4}}\, t\right)$$

$$\Rightarrow \frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4} = \lambda_n$$

$$0 = T_n'(t) \Rightarrow b_n = 0 .$$

$$\therefore \boxed{T_n(t) = \cos\left(\sqrt{\frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4}}\, t\right)}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t)}$$

$$x(1-x)(1+x)^{1/2} = u(x,0) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0) = (1+x)^{1/2} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ln(2)} \ln(1+x)\right)$$

$$b_n = \frac{\left\langle x(1-x)(1+x)^{1/2},\ (1+x)^{1/2}\sin\left(\frac{n\pi}{\ln(2)}\ln(1+x)\right)\right\rangle}{\left\langle (1+x)^{1/2}\sin\left(\frac{n\pi}{\ln(2)}\ln(1+x)\right),\ (1+x)^{1/2}\sin\left(\frac{n\pi}{\ln(2)}\ln(1+x)\right)\right\rangle}$$

$$\sin^2(n\pi u) = \frac{1-\cos(2\ldots}{2}$$

$$D = \int_0^1 \left[(1+x)^{1/2}\sin\left(\frac{n\pi}{\ln(2)}\ln(1+x)\right)\right]^2 (1+x)^{-2}\,dx$$

$$= \int_0^1 \sin^2\left(\frac{n\pi}{\ln(2)}\ln(1+x)\right)(1+x)^{-1}\,dx$$

$x=0 \Rightarrow u=0$

$x=1 \Rightarrow u=1$

Let $u = \frac{\ln(1+x)}{\ln(2)}$

Then $du = \frac{(1+x)^{-1}}{\ln(2)}dx$

$$= \int_0^1 \sin^2(n\pi u)\ln(2)\,du = \ln(2)\int_0^1 \frac{1-\cos(2n\pi u)}{2} \ldots \frac{\ln(2)}{\ldots}$$

$$\left\{\begin{array}{l} u=0: \quad -2+3-1=0 \\ u=1: \quad \cancel{} \\ \quad -2+3e^{\ln(2)}-e^{2\ln(2)} \\ \quad =-2+3(2)-4 \\ \quad \overset{\checkmark}{=} 0 \end{array}\right.$$

$$N = \int_0^1 (1+x)^{1/2} x(1-x)(1+x)^{1/2} \sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)(1+x)^{-2}\,dx$$

$$= \int_0^1 x(1-x)\sin\left(\frac{n\pi \ln(1+x)}{\ln(2)}\right)(1+x)^{-1}\,dx$$

Let $u = \dfrac{\ln(1+x)}{\ln(2)}$

Then $\ln(2)\,du = (1+x)^{-1}dx$

$$= \int_0^1 \underbrace{\left(e^{\ln(2)u}-1\right)\left(2-e^{\ln(2)u}\right)}_{-2+3e^{\ln(2)u}-e^{2\ln(2)u}} \sin(n\pi u)\ln(2)\,du$$

Also $\ln(2)u = \ln(1+x)$

$\Rightarrow \quad e^{\ln(2)u} = 1+x$

$\Rightarrow \quad e^{\ln(2)u} - 1 = x$

and $-(e^{\ln(2)u}-1)+1 = 1-x$

$2-e^{\ln(2)u} = 1-x$

$$\int e^{au}\sin(bu)\,du = \frac{e^{au}\left[a\sin(bu) - b\cos(bu)\right]}{a^2+b^2} + C$$

$$I_1 = -2\ln(2)\int_0^1 \sin(n\pi u)\,du = +2\ln(2)\frac{\cos(n\pi u)}{n\pi}\Big|_0^1$$

$$I_2 = 3\ln(2)\int_0^1 e^{\ln(2)u}\sin(n\pi u)\,du = \frac{3\ln(2)\,e^{\ln(2)u}}{\ln^2(2)+n^2\pi^2}\left[\ln(2)\sin(n\pi u) - n\pi\cos(n\pi u)\right]\Big|_0^1$$

$$I_3 = -\ln(2)\int_0^1 e^{2\ln(2)u}\sin(n\pi u)\,du = \frac{-\ln(2)\,e^{2\ln(2)u}}{4\ln^2(2)+n^2\pi^2}\left[2\ln(2)\sin(n\pi u) - n\pi\cos(n\pi u)\right]\Big|_0^1$$

$$\left\{\begin{array}{l} (2-\beta)(\beta-1) \\ = 2\beta - 2 - \beta^2 + \beta \\ = -\beta^2 + 3\beta - 2 \end{array}\right.$$

$$I_1 = \frac{2\ln(2)}{n\pi}\left[\cos(n\pi)-1\right] = \frac{2\ln(2)}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{-4\ln(2)}{n\pi} & \text{if } n \text{ is odd,} \end{cases}$$

$$I_2 = \frac{3\ln(2)}{\ln^2(2)+n^2\pi^2}\left[e^{\ln(2)}(-n\pi)\cos(n\pi) + n\pi\right] = \frac{3\ln(2)(n\pi)}{\ln^2(2)+n^2\pi^2}\left[1 - 2(-1)^n\right] = \begin{cases} \dfrac{-3\ln(2)(n\pi)}{\ln^2(2)+n^2\pi^2} & \text{if } n \text{ is ev} \\ \dfrac{9\ln(2)(n\pi)}{\ln^2(2)+n^2\pi^2} & \text{if } n \text{ is od} \end{cases}$$

$$I_3 = \frac{-\ln(2)}{4\ln^2(2)+n^2\pi^2}\left[e^{2\ln(2)}(-n\pi)\cos(n\pi) + n\pi\right] = \frac{-\ln(2)(n\pi)}{4\ln^2(2)+n^2\pi^2}\left[1 - (-1)^n 4\right] = \begin{cases} \dfrac{3\ln(2)(n\pi)}{4\ln^2(2)+n^2\pi^2} & \text{if } n \text{ is e} \\ -5\ln(2)(n\pi) & \text{if } n \text{ is} \end{cases}$$

$$-\frac{4}{x^2}+\frac{9}{b+x^2}-\frac{5}{4b+x^2}=\frac{9(4b+x^2)-5(b+x^2)}{(b+x^2)(4b+x^2)}-\frac{4}{x^2}$$

$$\frac{1}{4b+x^2}-\frac{1}{b+x^2}=\frac{b+\cancel{x^2}-(4b+\cancel{x^2})}{(a+x^2)(b+x^2)}$$

$$N=I_1+I_2+I_3$$

$$1=\begin{cases}3\ln(2)(n\pi)\left[\dfrac{1}{4\ln^2(2)+n^2\pi^2}-\dfrac{1}{\ln^2(2)+n^2\pi^2}\right] & \text{if } n \text{ is even}\\[2ex] \dfrac{-4\ln(2)(n\pi)}{(n\pi)^2}+\dfrac{9\ln(2)(n\pi)}{\ln^2(2)+n^2\pi^2}-\dfrac{5\ln(2)(n\pi)}{4\ln^2(2)+n^2\pi^2} & \text{if } n \text{ is odd}\end{cases}$$

$$=\begin{cases}\dfrac{-9\ln^3(2)(n\pi)}{(4\ln^2(2)+n^2\pi^2)(\ln^2(2)+n^2\pi^2)} & \text{if } n \text{ is even}\\[2ex] (n\pi)\ln(2)\left[\dfrac{-4}{(n\pi)^2}+\dfrac{9}{\ln^2(2)+n^2\pi^2}-\dfrac{5}{4\ln^2(2)+n^2\pi^2}\right] & \text{if } n \text{ is odd}\end{cases}$$

$$N=\begin{cases}\dfrac{-9\ln^3(2)(n\pi)}{(4\ln^2(2)+n^2\pi^2)(\ln^2(2)+n^2\pi^2)} & \text{if } n \text{ is even,}\\[2ex] \dfrac{(n\pi)\ln^3(2)\left[11n^2\pi^2-16\ln^2(2)\right]}{n^2\pi^2\left[\ln^2(2)+n^2\pi^2\right]\left[4\ln^2(2)+n^2\pi^2\right]} & \text{if } n \text{ is odd.}\end{cases}$$

$$\boxed{b_n=\frac{N}{D}=\frac{2N}{\ln(2)}=\begin{cases}\dfrac{-18\ln^2(2)(n\pi)}{(4\ln^2(2)+n^2\pi^2)(\ln^2(2)+n^2\pi^2)} & \text{if } n \text{ is even,}\\[2ex] \dfrac{2\ln^2(2)\left[11n^2\pi^2-16\ln^2(2)\right]}{n\pi\left[\ln^2(2)+n^2\pi^2\right]\left[4\ln^2(2)+n^2\pi^2\right]} & \text{if } n \text{ is odd}\end{cases}}$$

$$\frac{x^2\overbrace{(36b+9x^2-5b-5x^2)}^{31b+4x^2}}{x^2(b+x^2)(4b+x^2)}-\frac{\overbrace{4(b+x^2)(4b+x^2)}^{4b^2+5bx^2+x^4}}{x^2(b+x^2)(4b+x^2)}=\frac{\overbrace{31bx^2+\cancel{4x^4}-16b^2-20bx^2}^{11bx^2}-\cancel{4x^4}}{x^2(b+x^2)(4b+x^2)}$$

E III $(n = 18)$

$\mu = 78.9$
$\sigma = 16.2$

87 – 110	𝍸 I	6
73 – 86	𝍸	5
60 – 72	IIII	4
50 – 59	III	3
0 – 49		