Mathematics 325 Exam III Name: Dr. Grow

Summer 2004

NOTE: On the first problem of this exam you are required to show ALL the details of the solution of the appropriate eigenvalue problem. On all subsequent problems of this exam, you may omit the details of any eigenvalue computations; a correct statement of the associated eigenvalues and eigenfunctions will suffice.

1.(25 pts.) Consider the equation

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1 \text{ and } -\infty < t < \infty$$

with the boundary conditions

$$u_x(0,t) = 0 \text{ and } u(1,t) = 0 \quad \text{for } t \geq 0.$$

(a) What is the name of the method that should be used to solve this problem?
(b) Use this method to arrive at an associated eigenvalue problem.
(c) What are the eigenvalues for this problem?
(d) Show that the corresponding eigenfunctions are $\cos((n+\frac{1}{2})\pi x)$ where $n = 0,1,2,\ldots$
(e) Write a formal series expansion for a solution to this problem.

5 (a) $\boxed{\text{Separation of Variables}}$

(b) $u(x,t) = X(x)T(t)$ (nontrivial) leads to $T''(t)X(x) - X''(x)T(t) = 0$ and $X'(0)T(t) = 0 = X(1)T(t)$. Hence $\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$. Therefore

5 $\boxed{X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X(1) = 0}$ is the eigenvalue problem.

(c)-(d) Case $\lambda > 0$, say $\lambda = \beta^2$ where $\beta > 0$.

$X'' + \beta^2 X = 0 \Rightarrow X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$ and $X'(x) = -\beta c_1 \sin(\beta x) + \beta c_2 \cos(\beta x)$

$0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$. $0 = X(1) = c_1 \cos(\beta) \Rightarrow \beta_n = \frac{(2n+1)\pi}{2} = (n+\frac{1}{2})\pi$

5 $(n = 0,1,2,\ldots)$. Eigenvalues: $\lambda_n = \beta_n^2 = (n+\frac{1}{2})^2\pi^2$ $(n = 0,1,2,\ldots)$

5 Eigenfunctions: $X_n(x) = \cos((n+\frac{1}{2})\pi x)$

Case $\lambda = 0$: $X'' = 0 \Rightarrow X(x) = c_1 x + c_2$. $0 = X'(0) = c_1$ and $0 = X(1) = c_2$

No nontrivial solutions, so 0 is not an eigenvalue.

Case $\lambda < 0$, say $\lambda = -\beta^2$ where $\beta > 0$.

$X'' - \beta^2 X = 0 \Rightarrow X(x) = c_1\cosh(\beta x) + c_2 \sinh(\beta x)$ and $X'(x) = \beta c_1 \sinh(\beta x) + \beta c_2 \cosh$

$0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$. $0 = X(1) = c_1\cosh(\beta) \Rightarrow c_1 = 0$.

There are no nontrivial solutions, so there are no negative eigenvalues.

(over)

(e) $T_n'' + \lambda_n T_n = 0$

arbitrary constants

$$T_n'' + \left(n+\tfrac{1}{2}\right)^2\pi^2 T_n = 0 \quad \Rightarrow \quad T_n(t) = a_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi t\right) + b_n \sin\left(\left(n+\tfrac{1}{2}\right)\pi t\right)$$

$u_n(x,t) = X_n(x)\,T_n(t)$ solves the problem for each $n = 0, 1, 2, \dots$

By linearity and homogeneity,

5

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \left[a_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi t\right) + b_n \sin\left(\left(n+\tfrac{1}{2}\right)\pi t\right)\right] \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)}$$

is a formal series expansion for a solution to this problem.

2.(25 pts.) Solve $u_t - u_{xx} = 0$ for $0 < x < \pi$ and $0 < t < \infty$, with the boundary conditions $u_x(0,t) = 0$ and $u_x(\pi,t) = 0$ for $t \geq 0$, and the initial condition $u(x,0) = \cos^2(x)$ for $0 \leq x \leq \pi$.

$u(x,t) = X(x)T(t)$ (nontrivial) in ①-②-③ leads to $T'(t)X(x) - X''(x)T(t) = 0 \Rightarrow$

$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$ and $X'(0)T(t) = 0$ and $X'(\pi)T(t) = 0$. Thus

4 $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(\pi) = 0$ is the associated eigenvalue

4 problem. The eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunctions

4 are $X_n(x) = \cos(nx)$, where $n = 0, 1, 2, \ldots$ $T_n'(t) + \lambda_n T_n(t) = 0 \Rightarrow$

5 $T_n(t) = e^{-\lambda_n t} = e^{-n^2 t}$. Thus $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}$ solves

①-②-③. We seek coefficients $a_0, a_1, a_2, \ldots$ so ④ is satisfied; i.e. for all $0 \leq x \leq \pi$,

$$\frac{1}{2} + \frac{1}{2}\cos(2x) = \cos^2(x) = u(x,0) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$$

4 By inspection, $a_0 = \frac{1}{2} = a_2$ and all other $a_n = 0$. Thus

4 $$\boxed{u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2x)e^{-4t}}$$

solves ①-②-③-④.

3.(25 pts.) Consider the constant function $\phi(x)=1$ on the closed interval $[0,1]$.

4 (a) Show that the Fourier sine series of ϕ on $[0,1]$ is
$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}.$$

4 (b) Does the Fourier sine series of ϕ converge to ϕ in the mean square sense on $[0,1]$? Support your answer with appropriate reasons.

5 (c) For which x in $[0,1]$ does the Fourier sine series of ϕ converge pointwise to $\phi(x)$? Support your answer with appropriate reasons.

4 (d) Does the Fourier sine series of ϕ converge to ϕ uniformly on $[0,1]$? Support your answer with appropriate reasons.

4 (e) Use the results above to help find the sum of the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$.

4 (f) Use the results above to help find the sum of the series $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$.

(a) $\varphi(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ where $b_n = \frac{\langle \varphi, \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle} = 2\int_0^1 \varphi(x)\sin(n\pi x)\,dx =$

$$2\int_0^1 \sin(n\pi x)\,dx = -\frac{2\cos(n\pi x)}{n\pi}\Big|_0^1 = \frac{-2\cos(n\pi)+2}{n\pi} = \frac{-2(-1)^n+2}{n\pi} = \begin{cases} 0 & \text{if } n=2k \text{ is even} \\ \frac{4}{(2k+1)\pi} & \text{if } n=2k+1 \text{ is o} \end{cases}$$

$$\therefore\ \varphi(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}$$

(b) Since $\int_0^1 \varphi^2(x)\,dx = 1 < \infty$, Theorem 3 ensures that the Fourier sine series converges to φ in the mean-square sense on $[0,1]$.

(c) Since $\varphi(x)=1$ and $\varphi'(x)=0$ are continuous on $[0,1]$, Theorem 4(i) ensures that the Fourier sine series of φ converges pointwise to $\varphi(x)$ for all $0<x<1$. At the end points $x=0$ and $x=1$, the Fourier sine series clearly converges to 0, which is not equal to $\varphi(x)=1$ there.

(d) No, the Fourier sine series of φ does not converge uniformly to φ on $[0,1]$, because uniform convergence implies pointwise convergence on $[0,1]$ and the Fourier sine series is not pointwise convergent to $\varphi(x)$ at all x in $[0,1]$. (See part (c).)

(e) Take $x=\frac{1}{2}$ in the identity $1=\varphi(x)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}$ for all $0<x<1$.

(OVER)

Thus $\quad 1 = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi/2)}{2k+1} = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$

so $\quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \boxed{\frac{\pi}{4}}$.

(f) We apply Parseval's identity

$$\sum_{n=1}^{\infty} |b_n|^2 \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 \varphi^2(x)\,dx$$

to the function $\varphi(x) = 1$ on $[0,1]$. This yields

$$\sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k+1)}\right)^2 \cdot \frac{1}{2} = 1$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \boxed{\frac{\pi^2}{8}}$$

4.(25 pts.) Consider a metal rod of length 1, insulated along its sides but not at its ends, which is initially at temperature 100. Suddenly both ends are plunged into a bath of temperature zero.

(a) Write the partial differential equation, boundary conditions, and initial condition that model the temperature of the rod.

(b) What is the name of the method that should be used to solve this problem?

(c) Use this method to find a formula for the temperature $u = u(x,t)$ at position x in [0,1] and time $t > 0$. (Hint: You may find the results of problem 3 useful.)

5 (a) $$\begin{cases} u_t - k u_{xx} \overset{①}{=} 0 & \text{for } 0<x<1,\ 0<t<\infty, \\ u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t) & \text{for } t \geq 0, \\ u(x,0) \overset{④}{=} 100 & \text{for } 0<x<1 \end{cases}$$

5 (b) Separation of Variables.

(c) $u(x,t) = X(x)T(t)$ (nontrivial) in ①-②-③ leads to $T'(t)X(x) - kX''(x)T(t) = 0$

$\Rightarrow \dfrac{T'(t)}{kT(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$ and $X(0)T(t) = 0 = X(1)T(t)$. Therefore

$X'' + \lambda X = 0$, $X(0) = 0$, $X(1) = 0$ is the eigenvalue problem. We have

2 eigenvalues: $\lambda_n = (n\pi)^2$

2 eigenfunctions: $X_n(x) = \sin(n\pi x)$ $\quad (n = 1, 2, 3, \ldots)$

2 $T_n' + k\lambda_n T_n = 0 \Rightarrow T_n(t) = e^{-k\lambda_n t} = e^{-kn^2\pi^2 t}$.

2 Thus $\displaystyle\sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-kn^2\pi^2 t}$ solves ①-②-③. To solve ④ we need

$$100 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for } 0<x<1$$

1 From problem 3, $\displaystyle 100 = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$ for $0<x<1$.

2 Therefore $b_n = 0$ if $n = 2k$ is even and $b_n = \dfrac{400}{\pi(2k+1)}$ if $n = 2k+1$ is odd.

Thus

2 $$\boxed{u(x,t) = \frac{400}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x)}{2m+1} e^{-k(2m+1)^2\pi^2 t}}.$$

Bonus(25 pts.): (a) State (without proof) the maximum/minimum principle for solutions to Laplace's equation.

(b) Use the maximum/minimum principle to show that there is at most one solution to

$$u_{xx} + u_{yy} = xy \quad \text{in the region } x^2 + y^2 < 1$$

satisfying

$$u(x, y) = 1 \text{ at all points on the boundary } x^2 + y^2 = 1.$$

10 (a) If $u = u(x,y)$ is a solution to $u_{xx} + u_{yy} = 0$ in an open bounded region D of the plane and u is continuous on $\overline{D} = D \cup \partial D$ then

$$\max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} u(x,y) \quad \text{and} \quad \min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} u(x,y).$$

15 (b) Suppose the problem had two solutions $u = u_1(x,y)$ and $u = u_2(x,y)$. Then $v(x,y) = u_1(x,y) - u_2(x,y)$ is a solution to

$$v_{xx} + v_{yy} = 0 \quad \text{in } D: \ x^2+y^2<1$$

satisfying

$$v(x,y) = 0 \quad \text{on } \partial D: \ x^2+y^2=1.$$

By the maximum/minimum principle,

$$0 = \max_{(x,y)\in\partial D} v(x,y) = \max_{(x,y)\in\overline{D}} v(x,y) \quad \text{and} \quad 0 = \min_{(x,y)\in\partial D} v(x,y) = \min_{(x,y)\in\overline{D}} v(x,y).$$

Consequently $v(x,y) = 0$ for all $(x,y) \in \overline{D}: \ x^2+y^2 \le 1$. That is,

$u_1(x,y) = u_2(x,y)$ for all $x^2+y^2 \le 1$.

Math 325

Exam III

Summer 2004

$\mu = 80.5$

$\sigma = 26.5$

Distribution of Scores

118
113
112
110
102
90
87 ↑ A

75 B
73 B
65 B
64 C
60 B
59 C
44 D
35 D

Distribution of Grades

A	7
B	4
C	2
D	2
F	0

Sources of Exam III Problems:

#1	Sec. 4.2, #2
#2	Sec. 5.1, #6 similar to
#3	Sec. 5.1, Example 1 and Sec. 5.3, similar to #7, #12
#4	Sec. 4.1, Similar to #2
Bonus	Sec. 5.1, Similar to Lecture HW ⑧.