Mathematics 325 | Exam III | Name: Dr. Grow
Summer 2006

1.(35 pts.) (a) Show that the operator $T = -\frac{d^2}{dx^2}$ is hermitian on $V = \{f \in C^2[0,1] : f'(0) = 0 = f(1)\}$ equipped with the standard inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$.

(b) Find all the eigenvalues and corresponding eigenfunctions of T on V.

(c) Does the set of functions $\left\{\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\right\}_{n=0}^{\infty}$ form an orthogonal system on $[0,1]$ with the standard inner product? Justify your answer.

(d) Show that the Fourier series of $f(x) = 1 - x^2$ with respect to $\left\{\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\right\}_{n=0}^{\infty}$ on $[0,1]$ is

$$\sum_{n=0}^{\infty} \frac{32(-1)^n \cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\pi^3 (2n+1)^3}.$$

(e) Write the partial sum consisting of the first two terms of the above Fourier series for f. On the same coordinate axes, sketch the graph of this partial sum and the graph of f.

(f) Assume that for every x in $[0,1]$, $f(x) = 1 - x^2$ is equal to its Fourier series in part (d). Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

+6 (a) Let f and g belong to V. Then

$$\langle Tf, g\rangle = \int_0^1 -f''(x)\overline{g(x)}dx \overset{\text{Two integrations by parts}}{=} \left[f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right]\Big|_{x=0}^{1} - \int_0^1 f(x)\overline{g''(x)}\,dx.$$

But $f(1) = 0 = \overline{g(1)}$ and $\overline{g'(0)} = 0 = f'(0)$ so $\langle Tf, g\rangle = \langle f, Tg\rangle$; that is, T is hermitian on V.

+6 (b) Since $T = -\frac{d^2}{dx^2}$ is hermitian on V, all its eigenvalues are real numbers. In fact, since $-f(x)f'(x)\Big|_{x=0}^{1} = 0$ for all real-valued functions f in V, all the eigenvalues of T on V are positive, say $\lambda = \beta^2$. Then $Tf = \lambda f$ on V becomes $f''(x) + \beta^2 f(x) = 0$, $f'(0) = 0$, $f(1) = 0$. Then

$f(x) = A\cos(\beta x) + B\sin(\beta x)$ and $f'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x)$. $0 = f'(0) = \beta B$ $\Rightarrow B = 0$. $0 = f(1) = A\cos(\beta) \Rightarrow \beta = \beta_n = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi \quad (n = 0, 1, 2, \ldots)$.

Therefore the eigenvalues and eigenfunctions are $\boxed{\lambda_n = (n+\frac{1}{2})^2\pi^2 \text{ and } X_n(x) = \cos((n+\frac{1}{2})\pi x)}$

+6 (c) Yes, $\left\{\cos((n+\frac{1}{2})\pi x)\right\}_{n=0}^{\infty}$ is an orthogonal system on $[0,1]$ because they are eigenfunctions of a hermitian operator corresponding to the distinct eigenvalues $\lambda_n = (n+\frac{1}{2})^2\pi^2 \quad (n = 0, 1, 2, \ldots)$.

+6 (d) $\boxed{1 - x^2 \sim \sum_{n=0}^{\infty} c_n \cos((n+\frac{1}{2})\pi x)}$ +1 where $c_n = \dfrac{\langle 1-x^2, \cos((n+\frac{1}{2})\pi x)\rangle}{\langle \cos(n+\frac{1}{2})\pi x, \cos(n+\frac{1}{2})\pi x\rangle}$ +1 $(n = 0, 1, 2, \ldots)$

$$\langle \cos((n+\tfrac{1}{2})\pi x), \cos((n+\tfrac{1}{2})\pi x)\rangle = \int_0^1 \cos^2((n+\tfrac{1}{2})\pi x)\,dx = \int_0^1 \left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2(n+\tfrac{1}{2})\pi x)\right]dx =$$

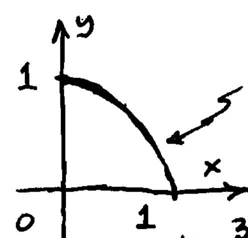
$$\left[\frac{x}{2} + \frac{1}{2(2n+1)\pi}\sin((2n+1)\pi x)\right]\Bigg|_0^1 = \frac{1}{2}.$$ +1

$$\langle 1-x^2, \cos((n+\tfrac{1}{2})\pi x)\rangle = \int_0^1 \underbrace{(1-x^2)}_{v}\underbrace{\cos((n+\tfrac{1}{2})\pi x)dx}_{dV} = \cancel{\frac{(1-x^2)\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\Bigg|_0^1} \to 0 - \int_0^1 \underbrace{(-2x)}_{v}\underbrace{\frac{\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}d}_{dV}$$

$$= \cancel{\frac{2x(-\cos((n+\frac{1}{2})\pi x))}{(n+\frac{1}{2})^2\pi^2}\Bigg|_0^1} \to 0 - \int_0^1 \frac{-\cos((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})^2\pi^2}\,2dx = \frac{2\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})^3\pi^3}\Bigg|_0^1 = \frac{2(-1)^n}{(n+\frac{1}{2})^3\pi^3}$$

$$\therefore \boxed{c_n = 2\cdot\frac{2(-1)^n}{\left[\frac{1}{2}(2n+1)\right]^3\pi^3} = \frac{32(-1)^n}{(2n+1)^3\pi^3}}$$ +3

+5 (e)



The graphs of $y = 1 - x^2$ and $y = \frac{32}{\pi^3}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{27}\cos\left(\frac{3\pi x}{2}\right)\right]$ are nearly indistinguishable on $[0,1]$

+6 (f) Assume $1 - x^2 = \sum_{n=0}^{\infty} \dfrac{32(-1)^n\cos((n+\frac{1}{2})\pi x)}{\pi^3(2n+1)^3}$ for all $0 \le x \le 1$. Set $x = 0$ to get $1 = \sum_{n=0}^{\infty}\dfrac{32(-1)^n}{\pi^3(2n+1)^3}$ so $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^3} = \boxed{\dfrac{\pi^3}{32}}$.

2.(35 pts.) Find a solution to

$$u_{tt}-u_{xx}=0 \quad \text{if} \quad 0<x<1,\ 0<t<\infty, \quad ①$$

which satisfies

$$u_x(0,t)=0=u(1,t) \quad \text{if} \quad t\ge 0, \quad ②\ ③$$

and

$$u(x,0)=1-x^2,\ u_t(x,0)=0 \quad \text{if} \quad 0\le x\le 1. \quad ⑤\ ④$$

(Hint: You may find the results of problem 1 useful.)

Bonus (15 pts.): Show that the solution to the problem above is unique.

We use separation of variables. We seek nontrivial solutions to ①-②-③-④ of the form $u(x,t)=X(x)T(t)$. Substituting this form into ①-②-③-④ yields

6 pts. to here

$$\begin{cases} \boxed{X''(x)+\lambda X(x)=0, \quad X'(0)=0=X(1),} \leftarrow \text{Eigenvalue Problem} \\ T''(t)+\lambda T(t)=0, \quad T'(0)=0. \end{cases}$$

12 pts. to here.

By #1, the eigenvalues and eigenfunctions are $\lambda_n=(n+\frac{1}{2})^2\pi^2$ and $X_n(x)=\cos((n+\frac{1}{2})\pi x)$ for $n=0,1,2,\dots$ The solution to the t-problem corresponding to $\lambda=\lambda_n$ is (up to a constant factor) $T_n(t)=\cos((n+\frac{1}{2})\pi t)$. Therefore $u(x,t)=\sum_{n=0}^{\infty} c_n\cos((n+\frac{1}{2})\pi x)\cos((n+\frac{1}{2})\pi t)$ is a formal solution to ①-②-③-④. We want to choose the coefficients to satisfy ⑤:

18 pts. to here

24 pts. to here.

$$1-x^2=u(x,0)=\sum_{n=0}^{\infty} c_n\cos((n+\tfrac{1}{2})\pi x) \quad \text{for all } 0\le x\le 1.$$

30 pts. to here.

By #1, $c_n=\dfrac{32(-1)^n}{\pi^3(2n+1)^3}$ for $n=0,1,2,\dots$ Thus the solution to ①-②-③-④-⑤ is

$$\boxed{u(x,t)=\sum_{n=0}^{\infty}\frac{32(-1)^n\cos((n+\frac{1}{2})\pi x)\cos((n+\frac{1}{2})\pi t)}{\pi^3(2n+1)^3}}.$$

35 pts. to here.

Bonus: We use energy methods to show that the solution is unique. Suppose there were another solution $u=v(x,t)$ to the problem. Then

3pts. to here. $w(x,t) = u(x,t) - v(x,t)$ would solve the problem ①-②-③-④ and

5 pts. to here $\quad$ (5') $\quad u(x,0) = 0 \quad$ if $\quad 0 \le x \le 1.$

Consider the energy function of w:

7 $$E(t) = \int_0^1 \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t) \right] dx .$$

Then
$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t) \right] dx$$

9 $$= \int_0^1 \left[w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx$$

$$\overset{\text{use ①}}{=} \int_0^1 \left[w_t(x,t) w_{xx}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx$$

$$= \int_0^1 \frac{\partial}{\partial x}\left[w_t(x,t) w_x(x,t) \right] dx$$

11 pts. to here
$$= w_t(x,t) w_x(x,t) \Big|_{x=0}^{1} .$$

But ③ and ④ yield $w_t(1,t) = 0$ and $w_x(0,t) = 0$ so $\frac{dE}{dt} = 0$.

13 Thus E is constant: $E(t) = E(0) = \int_0^1 \left[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0) \right] dx = 0$ by ④ and (5') for all $t \ge 0$. By the vanishing theorem, $w_t(x,t) = 0$ and $w_x(x,t) = 0$ for all $0 \le x \le 1$ and each fixed $t \ge 0$. Therefore

15 pts. to here. (5') implies $w(x,t) = 0$; i.e. $v(x,t) = u(x,t)$ and the solution obtained above is unique.

3.(30 pts.) Use the method of separation of variables to find a solution of the beam equation

$$u_{tt} + u_{xxxx} \overset{①}{=} 0 \quad \text{if} \quad 0 < x < 1,\ 0 < t < \infty,$$

which satisfies the boundary conditions

$$u(0,t) \overset{②}{=} u(1,t) \overset{③}{=} u_{xx}(0,t) \overset{④}{=} u_{xx}(1,t) \overset{⑤}{=} 0 \quad \text{if} \quad t \ge 0,$$

and the initial conditions

$$u(x,0) \overset{⑦}{=} 2\sin(\pi x) - 3\sin(5\pi x) \quad \text{and} \quad u_t(x,0) \overset{⑥}{=} 0 \quad \text{if} \quad 0 \le x \le 1.$$

We seek nontrivial solutions to ①-②-③-④-⑤-⑥ of the form $u(x,t) = X(x)T(t)$. Substituting into the PDE ① and the BC/ICs ②-⑥ leads to

6 pts to here

$$\begin{cases} \boxed{X^{(4)}(x) - \lambda X(x) = 0, \quad X(0) = X(1) = X''(0) = X''(1) = 0,} & \text{Eigenvalue Problem} \\ T''(t) + \lambda T(t) = 0,\ T'(0) = 0. \end{cases}$$

It is easy to check that the operator $\frac{d^4}{dx^4}$ is hermitian on $V = \{ f \in C^4[0,1] : f(0) = f(1) = f''(0) = f''(1) = 0 \}$, so the eigenvalues of the problem are real. In fact, if λ is an eigenvalue let $0 \ne X \in V$ such that $X^{(4)} = \lambda X$. Then two integrations by parts shows that

$$\lambda\langle X, X\rangle = \langle \lambda X, X\rangle = \langle X^{(4)}, X\rangle = \int_0^1 X^{(4)}(x)\overline{X(x)}\,dx = \left(\overline{X(x)}X^{(3)}(x) - \overline{X'(x)}X''(x)\right)\Bigg|_{x=0}^{1} + \int_0^1 X''(x)\overline{X''(x)}\,dx.$$

But $\overline{X(0)} = X''(0) = 0$ and $\overline{X}(1) = X''(1) = 0$ so $\lambda\langle X, X\rangle = \langle X'', X''\rangle \ge 0$.

10 Since $\langle X, X\rangle > 0$ it follows that $\lambda \ge 0$.

<u>Case $\lambda > 0$:</u> Let $\lambda = \alpha^4$ where $\alpha > 0$. The eigenvalue equation becomes $X^{(4)}(x) - \alpha^4 X(x) = 0$.

$X(x) = e^{rx}$ leads to $r^4 e^{rx} - \alpha^4 e^{rx} = 0 \Rightarrow r^4 - \alpha^4 = 0 \Rightarrow (r^2 - \alpha^2)(r^2 + \alpha^2) = 0$

$\Rightarrow r = \pm\alpha, \pm i\alpha$. Thus $X(x) = \tilde{c}_1 e^{\alpha x} + \tilde{c}_2 e^{-\alpha x} + \tilde{c}_3 e^{i\alpha x} + \tilde{c}_4 e^{-i\alpha x}$ is the general solution of the DE. Equivalently, $X(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x) + c_3\cos(\alpha x) + c_4\sin(\alpha x)$ and hence $X''(x) = \alpha^2 c_1\cosh(\alpha x) + \alpha^2 c_2\sinh(\alpha x) - \alpha^2 c_3\cos(\alpha x) - \alpha^2 c_4\sin(\alpha x)$.

$0 = X(0) = c_1 + c_3$ and $0 = X''(0) = \alpha^2 c_1 - \alpha^2 c_3$ imply $c_1 = c_3 = 0$. Then

$0 = X(1) = c_2\sinh(\alpha) + c_4\sin(\alpha)$ and $0 = X''(1) = \alpha^2 c_2\sinh(\alpha) - \alpha^2 c_4\sin(\alpha)$.

Adding these last two equations yields $0 = 2\alpha^2 c_2 \underbrace{\sinh(\alpha)}_{\text{positive}} \Rightarrow c_2 = 0$.

Then $0 = \alpha^2 c_4 \sin(\alpha)$ so $\sin(\alpha) = 0$ is the eigenvalue condition. Therefore

16 pts. to here $\alpha = \alpha_n = n\pi$ $(n = 1, 2, 3, \ldots)$ so $\lambda_n = \alpha_n^4 = (n\pi)^4$ and $X_n(x) = \sin(n\pi x)$ are the eigenvalues and eigenfunctions, respectively, where $n = 1, 2, 3, \ldots$

Case $\lambda = 0$: The eigenvalue equation becomes $X^{(4)} = 0$ so $X(x) = c_1x^3 + c_2x^2 + c_3x + c_4$ and $X''(x) = 6c_1x + 2c_2$. Then $0 = X(0) = c_4$ and $0 = X''(0) = 2c_2 \Rightarrow c_2 = 0$.

18 Also $0 = X(1) = c_1 + c_3$ and $0 = X''(1) = 6c_1 \Rightarrow c_1 = 0 = c_3$. Therefore, there is no nontrivial solution so zero is not an eigenvalue.

The solution of the t-equation $T_n''(t) + \lambda_n T_n(t) = 0 \Leftrightarrow T_n''(t) + (n\pi)^4 T_n(t) = 0$ is $T_n(t) = c_1\cos(n^2\pi^2t) + c_2\sin(n^2\pi^2t)$. Hence $T_n'(t) = -n^2\pi^2c_1\sin(n^2\pi^2t) + n^2\pi^2c_2\cos(n^2\pi^2t)$ so $0 = T_n'(0) = n^2\pi^2c_2 \Rightarrow c_2 = 0$. Thus $T_n(t) = \cos(n^2\pi^2t)$, up to a constant factor.

26 By the superposition principle, $u(x,t) = \sum_{n=1}^{N} c_n \sin(n\pi x)\cos(n^2\pi^2t)$ solves the homogeneous portion of the problem ①-②-③-④-⑤-⑥ for every integer $N \geq 1$ and all constants $c_1, \ldots, c_N$. We want to satisfy the inhomogeneous condition ⑦ so $2\sin(\pi x) - 3\sin(5\pi x) = u(x,0) = \sum_{n=1}^{N} c_n \sin(n\pi x)$ for all $0 \leq x \leq 1$.

Consequently we may take $N = 5$ and $c_1 = 2$, $c_5 = -3$, and other $c_n = 0$. That is,

30 pts. to here

$$\boxed{u(x,t) = 2\sin(\pi x)\cos(\pi^2 t) - 3\sin(5\pi x)\cos(25\pi^2 t)}$$

is a solution of the problem ①-②-③-④-⑤-⑥-⑦.

Note: Using the energy function $E(t) = \int_0^1 \left[\frac{1}{2}u_t^2(x,t) + \frac{1}{2}u_{xx}^2(x,t)\right]dx$, it can be shown that this solution is unique.

Math 325

Exam III

Summer 2006

$n = 15$

mean = 52.7

standard deviation = 20.6

Distribution of Scores:

	Graduate	Undergrad	Frequency
87 – 100	A	A	0
73 – 86	B	B	3
60 – 72	C	B	2
50 – 59	C	C	6
0 – 49	F	D	4