Mathematics 325 Final Exam Name: Dr. Grow

Fall 2005

1.(25 pts.) Solve the partial differential equation $\frac{1}{2}u_x + (4x + xy)u_y = 0$ subject to the auxiliary condition $u(0, y) = y^2 + 8y$ for $-\infty < y < \infty$.

2.(25 pts.) Find the general solution of $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} = \cos(x + y)$ in the xy-plane.

3.(25 pts.) Solve $u_t - u_{xx} = 0$ in $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition

$$u(x,0) = \phi(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Express your answer in terms of the error function:

$$Erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-p^2}\, dp.$$

4.(25 pts.) Solve $u_{tt} - u_{xx} = 0$ in $-\infty < x < \infty$, $-\infty < t < \infty$, subject to the initial conditions $u(x,0) = e^{-x^2}$ and $u_t(x,0) = -2xe^{-x^2}$ for $-\infty < x < \infty$.

5.(25 pts.) Use the Fourier transform method to find a formula for the solution to the inhomogeneous diffusion problem in the upper half-plane:

$$u_t - ku_{xx} = f(x,t) \quad \text{if } -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) = \phi(x) \quad \text{if } -\infty < x < \infty.$$

Notes: A. The solution is rumored to be

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(y,s) \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}}\, dy\, ds + \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}\, dy.$$

B. If you cannot solve this problem in its full generality, to earn half the points, do the special case when $f(x,t) \equiv 0$.

6.(25 pts.) (a) Show that the Fourier sine series of $f(x) = 3x^5 - 10x^3 + 7x$ on the unit interval $0 \le x \le 1$ is

$$\frac{720}{\pi^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n\pi x)}{n^5}.$$

(b) For which values of x in $[0,1]$ does the Fourier sine series of f converge pointwise to $f(x)$? Justify your answer.

(c) Does the Fourier sine series of f converge to f in the mean square sense on $[0,1]$? Why?

(d) Does the Fourier sine series of f converge uniformly to f on $[0,1]$? Why?

(e) Apply Parseval's identity to find the sum of $\sum_{n=1}^{\infty} \frac{1}{n^{10}}$.

7.(25 pts.) (a) Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the unit cube $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, given that $u_z(x, y, 1) = \cos(\pi x)\cos^3(\pi y)$ for $0 \le x \le 1$, $0 \le y \le 1$, and that u satisfies homogeneous Neumann boundary conditions on the other five faces of the cube. (Note: $4\cos^3(\theta) = 3\cos(\theta) + \cos(3\theta)$.)

(b) Is the solution to the problem in part (a) unique? Support your answer with reasons.

8.(25 pts.) (a) Use the method of separation of variables to find a solution of

$$u_{tt} + u_{xxxx} = 0 \quad \text{in the strip} \quad 0 < x < 1,\ 0 < t < \infty,$$

which satisfies the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \quad \text{for} \quad t \geq 0$$

and the initial conditions

$$u(x,0) = 3x^5 - 10x^3 + 7x \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq 1.$$

(Note: You may find useful the results of problem 6.)

(b) Is the solution to the problem in part (a) unique? Justify your answer.

Bonus (25 pts.): The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Celsius on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies $u_r = -\gamma$ where γ is a positive constant.

(a) Find the temperature distribution function for the material.

(b) What are the hottest and coldest temperatures in the material?

(c) Is it possible to choose γ so that the temperature on the outer boundary is 20 degrees Celsius? Support your answer with reasons.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x}$ $(a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and
(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.
(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

$\boxed{\#1.}$ $\frac{1}{2}u_x + (4+y)x u_y = 0$

$$\left\langle \tfrac{1}{2}, (4+y)x \right\rangle \cdot \nabla u = 0$$

6 Characteristics: $\dfrac{dy}{dx} = \dfrac{(4+y)x}{1/2}$

$$\int \frac{dy}{4+y} = \int 2x\,dx$$

$$\ln|4+y| = x^2 + c$$

$$y+4 = Ae^{x^2}$$

$$y = Ae^{x^2} - 4$$

12

Along a characteristic, the solution $u = u(x,y)$ has the value

$$u(x,y) = u(x, Ae^{x^2}-4) = u(0, A-4) = f(A)$$

where f is a C^1-function of a single real variable. Therefore

18 $$u(x,y) = f\left((y+4)e^{-x^2} \right)$$

is the general solution. We want

$y^2 + 8y = u(0,y) = f(y+4)$ for all $-\infty < y < \infty$. Let $z = y+4$.

Then

$$(z-4)^2 + 8(z-4) = f(z)$$

$$\Rightarrow \quad z^2 - \cancel{8z} + 16 + \cancel{8z} - 32 = f(z)$$

23 $$z^2 - 16 = f(z).$$

Thus, the particular solution is

$$u(x,y) = \left[(y+4)e^{-x^2} \right]^2 - 16$$

25 $$\Rightarrow \quad \boxed{u(x,y) = (y+4)^2 e^{-2x^2} - 16}$$

$\boxed{\#2}$ $u_{xx} - 4u_{xy} + 3u_{yy} = \cos(x+y)$ $B^2 - 4AC = 16 - 12 = 4 > 0$ (hyperbolic)

1 $$\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \cos(x+y)$$

8 Let $\begin{cases} \xi = +3x + y \\ \eta = +x + y . \end{cases}$ Then $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$

$\therefore$ $\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} = -2\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi}$. Therefore the p.d.e. is

12 equivalent to $-4\frac{\partial^2 u}{\partial \eta \partial \xi} = \cos(\eta) \Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = -\frac{1}{4}\cos(\eta) \Rightarrow \frac{\partial u}{\partial \xi} = \frac{-\sin(\eta)}{4} + c_1(\xi$

22 $$\Rightarrow u = \int \left[c_1(\xi) - \frac{\sin(\eta)}{4}\right] d\xi = A(\xi) - \frac{\xi}{4}\sin(\eta) + B(\eta).$$

Therefore, the general solution of the p.d.e. in the xy-plane is

25 $$\boxed{u(x,y) = f(y+3x) + g(x+y) - \left(\frac{y+3x}{4}\right)\sin(x+y)}$$

where f and g are C^2-functions of a single real variable.

$\boxed{\#3}$ $\begin{cases} u_t - u_{xx} = 0 & \text{if } -\infty<x<\infty,\ 0<t<\infty, \\ u(x,0) = \varphi(x) = \begin{cases} 1 & \text{if } x>0, \\ -1 & \text{if } x<0. \end{cases} \end{cases}$

$$\Rightarrow u(x,t) \overset{(6)}{=} \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy \overset{(12)}{=} \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}}(-1)\,dy + \frac{1}{\sqrt{4\pi t}}\int_{0}^{\infty} e^{-\frac{(x-y)^2}{4t}}(1)\,d$$

Let $p = \frac{y-x}{\sqrt{4t}}$. Then $dp = \frac{dy}{\sqrt{4t}}$.

(18) $$u(x,t) = \frac{-1}{\sqrt{\pi}}\int_{-\infty}^{\frac{-x}{\sqrt{4t}}} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{\frac{-x}{\sqrt{4t}}}^{\infty} e^{-p^2}\,dp$$

$$= -\left(\frac{1}{2} + \frac{1}{2}\mathrm{Erf}\left(\frac{-x}{\sqrt{4t}}\right)\right) + \left(\frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\frac{-x}{\sqrt{4t}}\right)\right)$$

(25) $$\boxed{u(x,t) = \mathrm{Erf}\left(\frac{x}{\sqrt{4t}}\right)}$$

$$\int_{-\infty}^{w} e^{-p^2}dp = \int_{-\infty}^{0} e^{-p^2}dp + \int_{0}^{w} e^{-p^2}dp$$

$$= \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2}\mathrm{Erf}(w)$$

$$\int_{w}^{\infty} e^{-p^2}dp = \int_{0}^{\infty} e^{-p^2}dp - \int_{0}^{w} e^{-p^2}dp$$

$$= \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}($$

$\boxed{\#4}$ $$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } -\infty < x < \infty,\ -\infty < t < \infty, \\ u(x,0) = e^{-x^2}, \quad u_t(x,0) = -2xe^{-x^2} & \text{for } -\infty < x < \infty. \end{cases}$$

(7) $$\therefore\ u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi \qquad \text{(d'Alembert's formula)}$$

(14) $$u(x,t) = \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} -2\xi e^{-\xi^2}\,d\xi$$

(21) $$= \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\, e^{-\xi^2}\Big|_{\xi=x-t}^{\xi=x+t}$$

$$= \frac{1}{2}\left[\cancel{e^{-(x-t)^2}} + e^{-(x+t)^2}\right] + \frac{1}{2}\left[e^{-(x+t)^2} - \cancel{e^{-(x-t)^2}}\right]$$

(25) $$\boxed{u(x,t) = e^{-(x+t)^2}}$$

$\boxed{\#5}$ $$\begin{cases} u_t - k u_{xx} = f(x,t) & \text{in} \quad -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \varphi(x) & \text{if} \quad -\infty < x < \infty. \end{cases}$$

$$\mathcal{F}(u_t)(\xi) - k\mathcal{F}(u_{xx})(\xi) = \mathcal{F}(f(x,t))(\xi) \equiv F(\xi,t)$$

($\mathcal{F}$ denotes the Fourier transform w.r.t. x)

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - k(i\xi)^2\mathcal{F}(u)(\xi) = F(\xi,t)$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2\mathcal{F}(u)(\xi) = F(\xi,t)$$

Integrating factor: $e^{\int k\xi^2 dt} = e^{k\xi^2 t}$.

$$e^{k\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}(u)(\xi) = e^{k\xi^2 t}F(\xi,t)$$

$$\frac{\partial}{\partial t}\left[e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right] = e^{k\xi^2 t}F(\xi,t)$$

$$e^{k\xi^2 t}\mathcal{F}(u)(\xi) = \int_0^t e^{k\xi^2\tau}F(\xi,\tau)\,d\tau + c_1(\xi)$$

Evaluating at $t=0$ and applying the initial condition yields

$$\mathcal{F}(\varphi)(\xi) = c_1(\xi).$$

$$\therefore \quad \mathcal{F}(u)(\xi) = e^{-k\xi^2 t}\int_0^t e^{k\xi^2\tau}F(\xi,\tau)\,d\tau + e^{-k\xi^2 t}\mathcal{F}(\varphi)(\xi)$$

$$= \int_0^t F(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau + e^{-k\xi^2 t}\mathcal{F}(\varphi)(\xi)$$

Apply Table entry I: $\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}$ with $a = \frac{1}{4kt}$

and obtain $\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi) = e^{-k\xi^2 t}$. Therefore

(17)
$$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,d\tau$$
$$+\ \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2kt}}\, e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$$

(18) But $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = \mathcal{F}\left(\frac{f*g}{\sqrt{2\pi}}\right)(\xi)$ so

(20)
$$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\left(f(\cdot,\tau) * \frac{1}{\sqrt{4\pi k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,d\tau + \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{(\cdot)^2}{4kt}}\right.$$

Interchanging the order of integration in the first term of the right member of the equation above yields

1)
$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\,d\tau\right)(\xi) + \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{(\cdot)^2}{4kt}}\right.$$

The inversion theorem then implies the desired result:

(25)
$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(y,\tau)\frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(x-y)^2}{4k(t-\tau)}}\,dy\,d\tau + \int_{-\infty}^{\infty} \varphi(y)\frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{(x-y)^2}{4kt}}\,d}$$

$\boxed{\#6}$ (a) The Fourier sine series of $f(x) = 3x^5 - 10x^3 + 7x$ on $[0,1]$ is

(8) $$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{where} \quad b_n = \frac{\langle f, \sin(n\pi \cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = 2\int_0^1 f(x)\sin(n\pi x)\,dx .$$ 2 pts. to here

Apply the repeated integration-by-parts formula:

4 pts. to here. $$\int_a^b f(x)g^{(5)}(x)\,dx = \Big[f(x)g^{(4)}(x) - f'(x)g'''(x) + f''(x)g''(x) - f'''(x)g'(x) + f^{(4)}(x)g(x)\Big]\Big|_a^b - \int_a^b f^{(5)}(x)g(x)\,dx$$

with $f(x) = 3x^5 - 10x^3 + 7x$, $g^{(5)}(x) = \sin(n\pi x)$, $a = 0$, and $b = 1$. Since the functions f, f'', and $f^{(4)}$ are odd, $f(0) = f''(0) = f^{(4)}(0) = 0$. One checks that $f(1) = f''(1) = f^{(4)}(1) = 0$ as well. Since the functions $g^{(5)}(x)$, $g^{(3)}(x)$, and $g'(x)$ are sine functions, we have

$$g^{(3)}(0) = g'(0) = 0 \quad \text{and} \quad g^{(3)}(1) = g'(1) = 0.$$ Consequently, all the "boundary terms" in the integration-by-parts vanish save $f^{(4)}(x)g(x)\Big|_0^1 = -360x\frac{\cos(n\pi x)}{(n\pi)^5}\Big|_0^1 = \frac{-360\cos(n\pi)}{(n\pi)^5}$

7 pts. to $$\text{Thus } b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\,\frac{(-360)\cos(n\pi x)}{(n\pi)^5} + \cancel{2\int_0^1 \frac{360\cos(n\pi x)}{(n\pi)^5}\,dx}^{\,0} = \frac{720(-1)^{n+1}}{(n\pi)^5},$$

and the Fourier sine series of f is

8 pts. to here $$\boxed{\sum_{n=1}^{\infty} \frac{720(-1)^{n+1}\sin(n\pi x)}{(n\pi)^5}}\ .$$

(4) (d) $$\left.\begin{aligned} f(x) &= 3x^5 - 10x^3 + 7x \\ f'(x) &= 15x^4 - 30x^2 + 7 \\ f''(x) &= 60x^3 - 60x \end{aligned}\right\} \text{These exist and are continuous functions on } 0 \le x \le 1.$$

Furthermore, $f(0) = 0 = f(1)$ so f satisfies the homogeneous Dirichlet boundary conditions $X(0) = 0 = X(1)$ that give rise to the sine eigenfunctions $X_n(x) = \sin(n\pi x)$ $(n = 1, 2, 3, \ldots)$ of $X'' + \lambda X = 0$ on $(0,1)$. Therefore, by Theorem 2, the Fourier sine series of f $\boxed{\text{converges uniformly}}$ to $f(x) = 3x^5 - 10x^3 + 7x$ on $[0,1]$.

(4) (b) Since uniform convergence implies pointwise convergence, part (d) shows that the Fourier sine series of f $\boxed{\text{converges pointwise}}$ to $f(x) = 3x^5 - 10x^3 + 7x$ for <u>every</u> x in $[0,1]$.

(4) (c) Since uniform convergence implies mean-square convergence, at least on bounded intervals, part (d) shows that the Fourier sine series of f $\boxed{\text{converges to } f \text{ in the mean-square sense}}$ on $[0,1]$.

(5) (e) $$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx \quad \text{so}$$ 1 pt. to here

$$\sum_{n=1}^{\infty} \left(\frac{720(-1)^{n+1}}{(n\pi)^5} \right)^2 \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 (3x^5 - 10x^3 + 7x)^2 dx$$ 2 pts. to here

$$\Rightarrow \quad \frac{(720)^2 \cdot \frac{1}{2}}{\pi^{10}} \sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{640}{231}$$ 4 pts. to here.

$$\Rightarrow \quad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}}$$ 5 pts. to here

$\boxed{\#7}$

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} \overset{①}{=} 0 \quad \text{in} \quad 0<x<1,\ 0<y<1,\ 0<z<1 \\ \left.\begin{array}{l} u_z(x,y,1) \overset{⑦}{=} \frac{3}{4}\cos(\pi x)\cos(\pi y) + \frac{1}{4}\cos(\pi x)\cos(3\pi y) \\ u_z(x,y,0) \overset{⑥}{=} 0 \end{array}\right\} \text{ for } 0<x<1,\ 0<y<1 \\ u_x(0,y,z) \overset{②}{=} 0 \overset{③}{=} u_x(1,y,z) \quad \text{for} \quad 0<y<1,\ 0<z<1 \\ u_y(x,0,z) \overset{④}{=} 0 \overset{⑤}{=} u_y(x,1,z) \quad \text{for} \quad 0<x<1,\ 0<z<1 \end{cases}$$

We seek nontrivial solutions to ①-⑥ of the form $u(x,y,z) = X(x)Y(y)Z(z)$.

$$① \Rightarrow \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0 \quad \text{so} \quad -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \lambda$$

and $-\dfrac{Y''(y)}{Y(y)} = \dfrac{Z''(z)}{Z(z)} - \lambda = \mu$. Applying ②-⑥ also yields

6 pts to here.

$$\left.\begin{array}{l} X''(x) + \lambda X(x) = 0,\ X'(0) = 0 = X'(1) \\ Y''(y) + \mu Y(y) = 0,\ Y'(0) = 0 = Y'(1) \\ Z''(z) - (\lambda+\mu)Z(z) = 0,\ Z'(0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} X_\ell(x) = \cos(\ell\pi x),\ \lambda_\ell = \ell^2\pi^2 \ (\ell=0, \\ Y_m(y) = \cos(m\pi y),\ \mu_m = m^2\pi^2 \ (m=0, \\ Z_{\ell,m}(z) = \cosh\left(\pi z\sqrt{\ell^2+m^2}\right) \end{array}$$

11 pts. to her

A formal solution to ①-⑥ is

$$u(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} c_{\ell,m}\cos(\ell\pi x)\cos(m\pi y)\cosh\left(\pi z\sqrt{\ell^2+m^2}\right).$$

15 pts. to here.

$$\therefore \quad u_z(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} c_{\ell,m}\pi\sqrt{\ell^2+m^2}\cos(\ell\pi x)\cos(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right)$$

16 pts. to h

We want to satisfy ⑦ so

$$⑦' \quad \frac{3}{4}\cos(\pi x)\cos(\pi y) + \frac{1}{4}\cos(\pi x)\cos(3\pi y) = \overbrace{\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} c_{\ell,m}\pi\sqrt{\ell^2+m^2}\sinh\left(\pi\sqrt{\ell^2+m^2}\right)\cos(\ell\pi x)\cos(m}^{u_z(x,y,1)}$$

for all $0<x<1,\ 0<y<1$.

17 pts. to here

Therefore we need

$$\frac{3}{4} = c_{1,1}\,\pi\sqrt{2}\,\sinh(\pi\sqrt{2})$$

$$\frac{1}{4} = c_{1,3}\,\pi\sqrt{10}\,\sinh(\pi\sqrt{10})$$

and all other $c_{\ell,m} = 0$, except $c_{0,0}$ which has no constraints placed upon it. (It is multiplied by 0 in the equation (7').)

21 pts to here.

$$\therefore \boxed{u(x,y,z) = c_{0,0} + \frac{3\cos(\pi x)\cos(\pi y)\cosh(\pi z\sqrt{2})}{4\pi\sqrt{2}\,\sinh(\pi\sqrt{2})} + \frac{\cos(\pi x)\cos(3\pi y)\cosh(\pi z\sqrt{10})}{4\pi\sqrt{10}\,\sinh(\pi\sqrt{10})}}$$

(b) $\boxed{\text{No}}$, the solution to the problem in part (a) is not unique. The constant $c_{0,0}$ is completely arbitrary.

4 pts.

$\boxed{\#8.}$ (a) $$\begin{cases} u_{tt} + u_{xxxx} \overset{1}{=} 0 & \text{for } 0<x<1,\ 0<t<\infty \\ u(0,t) \overset{2}{=} 0 \overset{3}{=} u(1,t) \text{ and } u_{xx}(0,t) \overset{4}{=} 0 \overset{5}{=} u_{xx}(1,t) & \text{for } t \geq 0 \\ u_t(x,0) \overset{6}{=} 0 \text{ and } u(x,0) \overset{7}{=} 3x^5 - 10x^3 + 7x & \text{for } 0 \leq x \leq 1 \end{cases}$$

We seek nontrivial solutions of 1–6 of the form $u(x,t) = X(x)T(t)$.

$$1 \Rightarrow \quad \frac{T''(t)}{T(t)} = -\frac{X^{(4)}(x)}{X(x)} = \lambda \ .$$ Also applying 2–6 yields

4 pts. to here. $$\begin{cases} X^{(4)}(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(1) \text{ and } X''(0) = 0 = X''(1) \\ T''(t) - \lambda T(t) = 0, \quad T'(0) = 0 \end{cases}$$

It is easy to see that the operator $L = -\frac{d^4}{dx^4}$ is self-adjoint on the vector space $\{\varphi \in C^4[0,1] : \varphi(0) = 0 = \varphi(1) \text{ and } \varphi''(0) = 0 = \varphi''(1)\} \equiv V$. (For suppose f and g belong to V. Then integration-by-parts shows

$$\langle Lf, g\rangle = \int_0^1 -f^{(4)}(x)\overline{g(x)}\,dx$$
$$= \left[-\overset{0}{\cancel{\overline{g(x)}}} f^{(3)}(x) + \overline{g'(x)} \overset{0}{\cancel{f''(x)}} - \overset{0}{\cancel{\overline{g''(x)}}} f'(x) + \overline{g^{(3)}(x)} \overset{0}{\cancel{f(x)}} \right]\Bigg|_0^1 + \int_0^1 f(x)\overline{\left[-g^{(4)}(x)\right]}$$
$$= \langle f, Lg\rangle . \Big)$$

Consequently, the eigenvalues of L on V are all real numbers.

(5) <u>Case $\lambda < 0$ (say $\lambda = -\beta^4$ where $\beta > 0$):</u> $X^{(4)}(x) - \beta^4 X(x) = 0$ has general solution

$X(x) = c_1\cos(\beta x) + c_2\sin(\beta x) + c_3\cosh(\beta x) + c_4\sinh(\beta x)$. Then $0 = X(0) = c_1 + c_3$.

$X''(x) = -\beta^2 c_1\cos(\beta x) - \beta^2 c_2 \sin(\beta x) + \beta^2 c_3\cosh(\beta x) + \beta^2 c_4 \sinh(\beta x)$. Then $0 = X''(0) = -\beta^2(c_1 - c_3)$

It follows that $c_1 = c_3 = 0$. Also $0 = X(1) = c_2\sin(\beta) + c_4\sinh(\beta)$

and $0 = \dfrac{X''(1)}{\beta^2} = -c_2\sin(\beta) + c_4\sinh(\beta)$.

Therefore $2c_4 \sinh(\beta) = 0 \Rightarrow c_4 = 0$ (since $\sinh(\beta) > 0$ for $\beta > 0$)

and $2c_2\sin(\beta) = 0$. In order for nontrivial solutions we must have $c_2 \neq 0$ and hence $\sin(\beta) = 0 \Rightarrow \beta = \beta_n = n\pi$ $(n = 1, 2, 3, \ldots)$. Thus, the negative eigenvalues

9 pts. to here. are $\lambda_n = -(n\pi)^4$ and the corresponding eigenfunctions are $X_n(x) = \sin(n\pi x)$ $(n = 1, 2, 3, \ldots)$.

(1) <u>Case $\lambda = 0$</u>: $X^{(4)}(x) = 0$ has general solution $X(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Then $X''(x) = 2c_2 + 6c_3 x$, so $0 = X(0) = c_0$ and $0 = X''(0) = 2c_2 \Rightarrow c_0 = 0 = c_2$. $\therefore X(x) = c_1 x + c_3 x^3$ and $X''(x) = 6c_3 x$. Consequently $0 = X(1) = c_1 + c_3$ and $0 = X''(1) = 6c_3 \Rightarrow c_1 = c_3 = 0$. There are no nontrivial solutions in this case;

10 " to here. i.e. zero is not an eigenvalue.

(5) <u>Case $\lambda > 0$ (say $\lambda = \beta^2$ where $\beta > 0$)</u>: $X^{(4)}(x) + \beta^4 X(x) = 0$ has general solution

$$X(x) = e^{\frac{\beta x}{\sqrt{2}}}\left(c_1\cos\left(\frac{\beta x}{\sqrt{2}}\right) + c_2\sin\left(\frac{\beta x}{\sqrt{2}}\right)\right) + e^{-\frac{\beta x}{\sqrt{2}}}\left(c_3\cos\left(\frac{\beta x}{\sqrt{2}}\right) + c_4\sin\left(\frac{\beta x}{\sqrt{2}}\right)\right).$$

To see this, $X(x) = e^{mx}$ in $X^{(4)} + \beta^4 X = 0$ leads to $m^4 + \beta^4 = 0 \Rightarrow m = \beta\sqrt[4]{-1}$. The fourth roots of -1 are of the form $\left(e^{\pi i + 2k\pi i}\right)^{1/4} = e^{\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)i}$ where $k = 0, 1, 2, 3$. That is, the values for m are

$$m_0 = \beta e^{\frac{i\pi}{4}} = \beta\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right), \quad m_1 = \beta e^{\frac{3\pi i}{4}} = \beta\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$m_2 = \beta e^{\frac{5\pi i}{4}} = \beta\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right), \quad m_3 = \beta e^{\frac{7\pi i}{4}} = \beta\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right).$$

It follows that $e^{m_0 x} = e^{\frac{\beta x}{\sqrt{2}}} \cdot e^{\frac{i\beta x}{\sqrt{2}}} = e^{\frac{\beta x}{\sqrt{2}}}\left(\cos\left(\frac{\beta x}{\sqrt{2}}\right) + i\sin\left(\frac{\beta x}{\sqrt{2}}\right)\right)$. Taking the real and imaginary parts of this complex solution yield the solutions $e^{\frac{\beta x}{\sqrt{2}}}\cos\left(\frac{\beta x}{\sqrt{2}}\right)$ and $e^{\frac{\beta x}{\sqrt{2}}}\sin\left(\frac{\beta x}{\sqrt{2}}\right)$. Similar computations with $e^{m_1 x}$ yield the real solutions $e^{-\frac{\beta x}{\sqrt{2}}}\cos\left(\frac{\beta x}{\sqrt{2}}\right)$ and $e^{-\frac{\beta x}{\sqrt{2}}}\sin\left(\frac{\beta x}{\sqrt{2}}\right)$.

A routine calculation shows that

$$X''(x) = e^{\frac{\beta x}{\sqrt{2}}}\left(-\beta^2 c_1 \sin\left(\frac{\beta x}{\sqrt{2}}\right) + \beta^2 c_2 \cos\left(\frac{\beta x}{\sqrt{2}}\right)\right) + e^{-\frac{\beta x}{\sqrt{2}}}\left(-\beta^2 c_3 \sin\left(\frac{\beta x}{\sqrt{2}}\right) - \beta^2 c_4 \cos\left(\frac{\beta x}{\sqrt{2}}\right)\right).$$

Then $0 = X(0) = c_1 + c_3$ and $0 = X''(0) = \beta^2(c_2 - c_4) \Rightarrow c_3 = -c_1$ and $c_4 = c_2$.

Thus $X(x) = c_1\left(e^{\frac{\beta x}{\sqrt{2}}} - e^{-\frac{\beta x}{\sqrt{2}}}\right)\cos\left(\frac{\beta x}{\sqrt{2}}\right) + c_2\left(e^{\frac{\beta x}{\sqrt{2}}} + e^{-\frac{\beta x}{\sqrt{2}}}\right)\sin\left(\frac{\beta x}{\sqrt{2}}\right)$

$$\Rightarrow \quad X(x) = 2c_1 \sinh\left(\frac{\beta x}{\sqrt{2}}\right)\cos\left(\frac{\beta x}{\sqrt{2}}\right) + 2c_2\cosh\left(\frac{\beta x}{\sqrt{2}}\right)\sin\left(\frac{\beta x}{\sqrt{2}}\right),$$

and similarly, $X''(x) = -2\beta^2 c_1 \cosh\left(\frac{\beta x}{\sqrt{2}}\right)\sin\left(\frac{\beta x}{\sqrt{2}}\right) + 2\beta^2 c_2 \sinh\left(\frac{\beta x}{\sqrt{2}}\right)\cos\left(\frac{\beta x}{\sqrt{2}}\right)$. Then

$$(*) \qquad 0 = \frac{X(1)}{2} = c_1 \sinh\left(\frac{\beta}{\sqrt{2}}\right)\cos\left(\frac{\beta}{\sqrt{2}}\right) + c_2\cosh\left(\frac{\beta}{\sqrt{2}}\right)\sin\left(\frac{\beta}{\sqrt{2}}\right) \qquad \text{and}$$

$$(**) \qquad 0 = \frac{X''(1)}{2\beta^2} = -c_1\cosh\left(\frac{\beta}{\sqrt{2}}\right)\sin\left(\frac{\beta}{\sqrt{2}}\right) + c_2\sinh\left(\frac{\beta}{\sqrt{2}}\right)\cos\left(\frac{\beta}{\sqrt{2}}\right).$$

Multiplying $(*)$ by $\cosh\left(\frac{\beta}{\sqrt{2}}\right)\sin\left(\frac{\beta}{\sqrt{2}}\right)$ and $(**)$ by $\sinh\left(\frac{\beta}{\sqrt{2}}\right)\cos\left(\frac{\beta}{\sqrt{2}}\right)$ and adding the results yields

$$\begin{aligned} 0 &= c_2\left[\cosh^2\left(\frac{\beta}{\sqrt{2}}\right)\sin^2\left(\frac{\beta}{\sqrt{2}}\right) + \sinh^2\left(\frac{\beta}{\sqrt{2}}\right)\cos^2\left(\frac{\beta}{\sqrt{2}}\right)\right] \\ &= c_2\left[\left(1 + \cancel{\sinh^2\left(\frac{\beta}{\sqrt{2}}\right)}\right)\sin^2\left(\frac{\beta}{\sqrt{2}}\right) + \sinh^2\left(\frac{\beta}{\sqrt{2}}\right)\left(1 - \cancel{\sin^2\left(\frac{\beta}{\sqrt{2}}\right)}\right)\right] \\ &= c_2\Big[\underbrace{\sin^2\left(\frac{\beta}{\sqrt{2}}\right) + \sinh^2\left(\frac{\beta}{\sqrt{2}}\right)}_{\text{positive for } \beta > 0}\Big]. \end{aligned}$$

$\therefore$ $c_2 = 0$. A similar argument shows $0 = c_1\left[\sinh^2\left(\frac{\beta}{\sqrt{2}}\right) + \sin^2\left(\frac{\beta}{\sqrt{2}}\right)\right]$ so $c_1 = 0$ as well.

Thus, there are no positive eigenvalues.

pts. to

ere.

Returning to the t-equation with $\lambda = \lambda_n = -(n\pi)^4$ $(n=1,2,3,\dots)$ yields

$$T_n''(t) + (n\pi)^4 T_n(t) = 0, \quad T_n'(0) = 0.$$

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The general solution is $T_n(t) = a\cos(n^2\pi^2 t) + b\sin(n^2\pi^2 t)$, and the initial condition implies $b=0$. Therefore, up to a constant factor, $T_n(t) = \cos(n^2\pi^2 t)$.

A formal solution of ①-⑥ is therefore

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(n^2\pi^2 t).$$

In order to satisfy ⑦ we need

$$3x^5 - 10x^3 + 7x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad \text{for} \quad 0 \le x \le 1.$$

Therefore, we need to choose $b_n = n^{th}$ Fourier sine coefficient of $f(x) = 3x^5 - 10x^3 + 7x$ on $[0,1]$. By problem #6, $b_n = \dfrac{720(-1)^{n+1}}{(n\pi)^5}$ $(n=1,2,3,\dots)$. Thus

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$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{720(-1)^{n+1}\sin(n\pi x)\cos(n^2\pi^2 t)}{(n\pi)^5}}$$

solves ①-⑦.

(6) (b) Yes, the solution to #8(a) is unique. To see this we apply energy methods. Let $v(x,t)$ be another solution to ①-⑦ in #8(a). Then $w(x,t) = u(x,t) - v(x,t)$ solves

$$\begin{cases} w_{tt} + w_{xxxx} \overset{①}{=} 0 & \text{in } 0<x<1,\ 0<t<\infty \\ w(0,t) \overset{②}{=} 0 \overset{③}{=} w(1,t) \text{ and } w_{xx}(0,t) \overset{④}{=} 0 \overset{⑤}{=} w_{xx}(1,t) & \text{for } t \ge 0, \\ w_t(x,0) \overset{⑥}{=} 0 \overset{⑦}{=} w(x,0) & \text{for } 0 \le x \le 1. \end{cases}$$

Let $E(t) = \int_0^1 \left[\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_{xx}^2(x,t) \right] dx$ denote the energy of the solution $w = w(x,t)$ at time $t \geq 0$. Then

$$E'(t) = \int_0^1 \frac{\partial}{\partial t} \left[\frac{1}{2} w_t^2 + \frac{1}{2} w_{xx}^2 \right] dx$$

$$= \int_0^1 \left[w_t w_{tt} + \overbrace{w_{xx}}^{U} \overbrace{w_{xxt}}^{dV} \right] dx$$

$$= \int_0^1 w_t w_{tt}\, dx + \cancelto{0 \text{ by } ④ \& ⑤}{w_{xx} w_{xt} \Big|_{x=0}^{1}} - \int_0^1 \overbrace{w_{xxx}}^{U} \overbrace{w_{xt}\, dx}^{dV}$$

$$= \int_0^1 w_t w_{tt}\, dx - w_{xxx} w_t \Big|_{x=0}^{1} + \int_0^1 w_{xxxx} w_t\, dx$$

$$= \int_0^1 w_t \cancelto{0 \text{ by } ①}{\left[w_{tt} + w_{xxxx} \right]} dx - w_{xxx}(1,t) \cancelto{0 \text{ as a consequence of } ③}{w_t(1,t)} + w_{xxx}(0,t) \cancelto{0 \text{ as a consequence of } ②}{w_t(0,t)}$$

$$= 0 .$$

Thus $E(t) = \text{constant} = E(0) = \int_0^1 \left[\frac{1}{2} \cancelto{0 \text{ by } ⑥}{w_t^2(x,0)} + \frac{1}{2} \cancelto{0 \text{ as a consequence of } ⑦}{w_{xx}^2(x,0)} \right] dx = 0.$

The vanishing theorem then implies $w_t(x,t) = 0 = w_{xx}(x,t)$ for all $0 \leq x \leq 1$, $0 \leq t < \infty$.

It follows that there exist constants a and b such that $w(x,t) = a + bx$ for all $0 \leq x \leq 1$, $0 \leq t < \infty$. But ② and ③ show that $a = 0 = b$; i.e. $w(x,t) \equiv 0$.

Thus $v(x,t) = u(x,t)$ for all $0 \leq x \leq 1$, $0 \leq t < \infty$, and uniqueness is demonstrated.

$\boxed{\text{Bonus}}$ Let $u = u(x,y,z)$ denote the steady-state temperature at position (x,y,z) in the shell $1 < x^2+y^2+z^2 < 4$. Then $u = u(x,y,z)$ satisfies

$$\begin{cases} \nabla^2 u \overset{①}{=} 0 & \text{in } 1 < x^2+y^2+z^2 < 4, \\ u(x,y,z) \overset{②}{=} 100 & \text{if } 1 = x^2+y^2+z^2, \\ \dfrac{\partial u}{\partial n}(x,y,z) \overset{③}{=} -\gamma & \text{if } 4 = x^2+y^2+z^2. \end{cases}$$

Normal derivative

We assume that $u = u(r)$, independent of θ and φ, based on the rotational invariance of the p.d.e, region, and boundary conditions. Therefore ① becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \cancelto{0}{\frac{\partial u}{\partial \varphi}}\right) + \frac{1}{r^2 \sin^2\varphi}\cancelto{0}{\frac{\partial^2 u}{\partial \theta^2}} = 0 .$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0 \quad \Rightarrow \quad r^2 \frac{\partial u}{\partial r} = c_1 \quad \Rightarrow \quad \frac{\partial u}{\partial r} = \frac{c_1}{r^2}$$

$\Rightarrow \; u = -\dfrac{c_1}{r} + c_2$. By ②, $100 = u(1) = -c_1 + c_2$.

By ③, $-\gamma = \left.\dfrac{\partial u}{\partial r}\right|_{r=2} = \dfrac{c_1}{2^2}$. Therefore $c_1 = -4\gamma$ and $c_2 = 100 - 4\gamma$

(12) so (a) $\boxed{u(r) = \dfrac{4\gamma}{r} + 100 - 4\gamma}$.

(6) (b) Since $u(r)$ is a decreasing function of r, the hottest temperature in the material is $u(1) = \boxed{100}$ and the coldest temperature is $u(2) = \boxed{100 - 2\gamma}$.

(c) $\quad 20 = u(2) = \dfrac{4\gamma}{2} + 100 - 4\gamma = 100 - 2\gamma$

(b)

$$\Rightarrow \gamma = \frac{100-20}{2} = \boxed{40}.$$

Yes, it is possible to choose γ so that the temperature on the outer boundary is 20 degrees Celsius.

Math 325
Fall 2005
Final Exam

Mean: 154.9

Standard Deviation: 40.5

n: 18

Distribution of Scores:	Graduate Grade / Undergraduate Grade	Frequency
174 or above	A / A	8
146 – 173	B / B	4
120 – 145	C / B	2
100 – 119	C / C	2
0 – 99	F / D	2