Mathematics 315 | Final Exam, Part I
Spring 2006 | Name: Dr. Grow
(1 pt.)

This portion of the 200 point final examination is "closed books/notes". You are to turn in your solutions to the problems on this portion before receiving the second part. The suggested maximum time to spend on this portion of the exam is 60 minutes.

1.(33 pts.) (a) State Lebesgue's Monotone Convergence Theorem.

(b) Give an example to show that the conclusion of the Monotone Convergence Theorem need not hold for pointwise increasing sequences of **negative** measurable functions.

(c) State Fatou's Lemma.

(d) Give an example to show that the inequality in the conclusion of Fatou's Lemma may actually be strict.

(e) State Lebesgue's Dominated Convergence Theorem.

(f) Use Fatou's Lemma to prove the Dominated Convergence Theorem.

2.(33 pts.) Let $f \in L^1[0,1]$.

(a) Show that $\left| \int_{[0,1]} f(x)dx \right| \le \int_{[0,1]} |f(x)| dx.$

(b) Show that $m\left(\{x \in [0,1] : |f(x)| \ge \lambda\}\right) \le \dfrac{\int_{[0,1]} |f(x)| dx}{\lambda}$ for all $\lambda > 0$.

(c) If $\int_{[0,1]} |f(x)| dx = 0$, show that $f(x) = 0$ a.e. in $[0,1]$.

(d) If $g(x) = f(x)$ a.e. in $[0,1]$, show that g is a measurable function on $[0,1]$, $\int_{[0,1]} |g(x)| dx < \infty$, and

$$\int_{[0,1]} g(x)dx = \int_{[0,1]} f(x)dx.$$

3.(33 pts.) In each of the following, compute the Lebesgue integral of f over the set E, or show that f is not integrable over E. Please justify the steps in your computations.

(a) $f(x) = \begin{cases} 3 & \text{if } x \in P \text{ (the Cantor set)}, \\ -1 & \text{if } x \in [0,1] \setminus P, \\ 2 & \text{if } x \in [-1,0] \setminus \mathbb{Q}, \\ -4 & \text{if } x \in [-1,0] \cap \mathbb{Q}. \end{cases}$ $\qquad E = [-1,1].$

(b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{A}, \\ \dfrac{1}{x} & \text{if } x \in \mathbb{R} \setminus \mathbb{A}. \end{cases}$ $\qquad E = [0,1].$

(c) $f(x) = \begin{cases} e^{x^2} & \text{if } x \in \mathbb{Q}, \\ \cos(x)e^{-x} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ $\qquad E = (0,\infty).$

(a) – (e) 5 pts. each, (f) 8 pts.

#1 (a) Let $f_n : E \to [0,\infty]$ $(n=1,2,3,\ldots)$ be a sequence of nonnegative measurable functions such that $f_1(x) \le f_2(x) \le f_3(x) \le \ldots$ for all x in E. Then $\int_E (\lim_{n\to\infty} f_n)\,dx = \lim_{n\to\infty} \int_E f_n\,dx$.

(b) Let $f_n(x) = -\frac{1}{x}\chi_{(n,\infty)}(x)$ for $n=1,2,3,\ldots$ and $x \in (1,\infty)$. Then $f_1(x) \le f_2(x) \le f_3(x) \le \ldots \le 0$ for all x in $(1,\infty)$. But $\lim_{n\to\infty} f_n(x) = 0$ on $(1,\infty)$ with $\int_1^\infty f_n\,dx = -\int_n^\infty \frac{1}{x}\,dx = -\infty < 0 = \int_1^\infty 0\,dx$ for all $n=1,2,3,\ldots$ so $\lim_{n\to\infty} \int_1^\infty f_n\,dx = -\infty < 0 = \int_1^\infty (\lim_{n\to\infty} f_n)\,dx$.

(c) Let $f_n : E \to [0,\infty]$ $(n=1,2,3,\ldots)$ be a sequence of nonnegative measurable functions. Then $\int_E (\liminf_{n\to\infty} f_n)\,dx \le \liminf_{n\to\infty} \int_E f_n\,dx$.

(d) Let $f_n = \chi_{(n,n+1)}$ for $n=1,2,3,\ldots$ Then $\lim_{n\to\infty} f_n(x) = 0$ for all x in $(0,\infty)$. But $\int_0^\infty f_n\,dx = 1$ for $n=1,2,3,\ldots$ and $\int_0^\infty 0\,dx = 0$ so $\lim_{n\to\infty} \int_0^\infty f_n\,dx = 1 > 0 = \int_0^\infty (\lim_{n\to\infty} f_n)\,dx$.

(e) Let $f_n : E \to [-\infty,\infty]$ $(n=1,2,3,\ldots)$ be a sequence of measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists in the extended real number sense for all x in E. If there exists $g \in L^1(E)$ such that $|f_n(x)| \le g(x)$ for all $n=1,2,3,\ldots$ and all $x \in E$, then $f \in L^1(E)$ and $\lim_{n\to\infty} \int_E f_n\,dx = \int_E f\,dx$.

(f) Proof: Clearly $f \in L^1(E)$. Let $h_n(x) = g(x) - f_n(x)$ for $n = 1, 2, 3, \ldots$ and $x \in E$. Then $\langle h_n \rangle$ is a sequence of nonnegative measurable functions on E and $\lim_{n\to\infty} h_n(x) = g(x) - f(x)$ on E. By Fatou's Lemma,

$$\int_E g\,dx - \int_E f\,dx = \int_E (g-f)\,dx \le \liminf_{n\to\infty} \int_E (g - f_n)\,dx$$

$$= \int_E g\,dx + \liminf_{n\to\infty}\left(-\int_E f_n\,dx\right)$$

$$= \int_E g\,dx - \limsup_{n\to\infty} \int_E f_n\,dx .$$

Since $0 \le \int_E g\,dx < \infty$, it follows that $\limsup_{n\to\infty} \int_E f_n\,dx \le \int_E f\,dx$.

Repeating this argument with $h_n = g + f_n$ $(n = 1, 2, 3, \ldots)$ we find that $\int_E f\,dx \le \liminf_{n\to\infty} \int_E f_n\,dx$. Therefore $\lim_{n\to\infty} \int_E f_n\,dx$ exists

and is equal to $\int_E f\,dx$.

(a) - (c) 7 pts. each (d) 12 pts.

#2 (a) $$\left|\int_{[0,1]} f\,dx\right| = \left|\int_{[0,1]} f^+dx - \int_{[0,1]} f^-dx\right| \le \int_{[0,1]} f^+dx + \int_{[0,1]} f^-dx$$

$$= \int_{[0,1]} (f^+ + f^-)dx = \int_{[0,1]} |f|\,dx.$$

(b) Let $\lambda > 0$ and $E_\lambda = \{x\in[0,1] : |f(x)| \ge \lambda\}$. Then $0 \le \lambda \chi_{E_\lambda} \le |f|$ on $[0,1]$ so $\lambda m(E_\lambda) = \int_{[0,1]} \lambda\chi_{E_\lambda}dx \le \int_{[0,1]} |f|\,dx$.

(c) Suppose $\int_{[0,1]} |f|\,dx = 0$. Then $m\left(\left\{x\in[0,1] : |f(x)| \ge \frac{1}{n}\right\}\right) = 0$ for $n = 1,2,3,\dots$ by part (b). But $\{x\in[0,1] : |f(x)| > 0\} = \bigcup_{n=1}^{\infty} \{x\in[0,1] : |f(x)| \ge \frac{1}{n}\}$ and $\{x\in[0,1] : |f(x)| \ge \frac{1}{n}\} \subseteq \{x\in[0,1] : |f(x)| \ge \frac{1}{n+1}\}$ for all $n = 1,2,3,\dots$,

so $$m\left(\{x\in[0,1] : |f(x)| > 0\}\right) = \lim_{n\to\infty} m\left(\left\{x\in[0,1] : |f(x)| \ge \frac{1}{n}\right\}\right) = 0.$$

That is, $f(x) = 0$ a.e. in $[0,1]$.

(d) Let $\lambda\in\mathbb{R}$. Then

$$\{x\in[0,1] : g(x) > \lambda\} = \{x\in[0,1] : f(x) = g(x) \text{ and } f(x) > \lambda\}$$
$$\cup\ \{x\in[0,1] : f(x) \ne g(x) \text{ and } g(x) > \lambda\}$$
$$\equiv A \cup B.$$

Note that $A = \{x \in [0,1] : f(x) = g(x)\} \cap \{x \in [0,1] : f(x) > \lambda\}$ is measurable since the first set in the RHS is the complement of a set of measure zero and hence is measurable while the second set in the RHS is measurable since f is a measurable function. Also $B = \{x \in [0,1] : f(x) \neq g(x) \text{ and } g(x) > \lambda\}$ is a subset of the measure zero set $\{x \in [0,1] : f(x) \neq g(x)\}$, and hence B is measurable. Consequently $\{x \in [0,1] : g(x) > \lambda\} = A \cup B$ is measurable, thus establishing that g is a measurable function.

Because $g = f$ a.e. if and only if both $g^+ = f^+$ a.e. and $g^- = f^-$ a.e., setting $E^+ = \{x \in [0,1] : g^+(x) = f^+(x)\}$ we have

$$\int_{[0,1]} g^+ dx = \int_{E^+} g^+ dx + \int_{[0,1]\setminus E^+} g^+ dx = \int_{E^+} f^+ dx + 0 = \int_{E^+} f^+ dx + \int_{[0,1]\setminus E^+} f^+ dx = \int_{[0,1]} f^+ dx.$$

A similar argument shows $\int_{[0,1]} g^- dx = \int_{[0,1]} f^- dx$. Therefore

$$\int_{[0,1]} |g|\, dx = \int_{[0,1]} g^+ dx + \int_{[0,1]} g^- dx = \int_{[0,1]} f^+ dx + \int_{[0,1]} f^- dx = \int_{[0,1]} |f|\, dx < \infty$$

and

$$\int_{[0,1]} g\, dx = \int_{[0,1]} g^+ dx - \int_{[0,1]} g^- dx = \int_{[0,1]} f^+ dx - \int_{[0,1]} f^- dx = \int_{[0,1]} f\, dx.$$

(a)-(c) 11 pts. each

#3 (a) Because $m(\mathbb{P}) = 0 = m(\mathbb{Q})$, $f = -1\chi_{[0,1]} + 2\chi_{[-1,0]}$ a.e. in $[-1,1]$.

By #2(d), $\int_{[-1,1]} f\,dx = \int_{[-1,1]} (-1\chi_{[0,1]} + 2\chi_{[-1,0]})dx = -1m([0,1]) + 2m([-1,0]) = \boxed{1}$.

(b) Since $\mathbb{A}$ is countable, $m(\mathbb{A}) = 0$. Therefore $f(x) = \frac{1}{x}$ a.e. in $\mathbb{R}$ so by #2(d), $\int_{[0,1]} f\,dx = \int_{[0,1]} \frac{1}{x}\,dx$. Let $f_n(x) = \min\{n, \frac{1}{x}\}$ for $n = 1, 2, 3, \ldots$ and x in $[0,1]$. Note that each f_n is continuous on $[0,1]$ and hence is a measurable function on $[0,1]$. Also $0 < f_1(x) \le f_2(x) \le \ldots$ for all $x \in [0,1]$ and $\lim_{n\to\infty} f_n(x) = \frac{1}{x}$ if $x \in [0,1]$. By the Monotone Convergence Theorem

$$\int_{[0,1]} \frac{1}{x}\,dx = \lim_{n\to\infty} \int_{[0,1]} f_n\,dx = \lim_{n\to\infty} \int_0^1 f_n\,dx = \lim_{n\to\infty} \left(\int_0^{1/n} n\,dx + \int_{1/n}^1 \frac{1}{x}\,dx \right)$$

$$= \lim_{n\to\infty} \left(1 + \ln(x)\Big|_{1/n}^{1} \right) = \lim_{n\to\infty} \big(1 + \ln(n) \big) = +\infty.$$

Therefore f is not integrable on $[0,1]$.

(c) Since $m(\mathbb{Q}) = 0$, $f(x) = \cos(x)e^{-x}$ a.e. in $\mathbb{R}$ so by #2(d), $\int_{(0,\infty)} f\,dx = \int_{(0,\infty)} \cos(x)e^{-x}\,dx$. Let $f_n(x) = \cos(x)e^{-x}\chi_{(0,n)}(x)$ for $n = 1, 2, 3, \ldots$ and $x \in (0,\infty)$. Each f_n is piecewise continuous and hence measurable on $(0,\infty)$. Also $\lim_{n\to\infty} f_n(x) = \cos(x)e^{-x}$ for all $x \in (0,\infty)$ and

$|f_n(x)| \leq e^{-x}$ for all $n = 1, 2, 3, \ldots$ and $x \in (0, \infty)$. Because the function $x \longmapsto e^{-x}$ is in $L^1(0, \infty)$, we may apply Lebesgue's Dominated Convergence Theorem to get

$$\int_{(0,\infty)} \cos(x) e^{-x} \, dx = \lim_{n\to\infty} \int_{(0,\infty)} f_n \, dx$$

$$= \lim_{n\to\infty} \left(\int_0^n e^{-x} \cos(x) \, dx \right)$$

$$= \lim_{n\to\infty} \left(\frac{\sin(x) - \cos(x)}{2e^x} \right) \Bigg|_0^n$$

$$= \lim_{n\to\infty} \left(\frac{\sin(n) - \cos(n)}{2e^n} - \frac{-1}{2} \right)$$

$$= \boxed{\frac{1}{2}} .$$

Computation of an integral:

$$\int \underbrace{\cos(x)}_{u} \underbrace{e^{-x} dx}_{dv} = -\cos(x) e^{-x} - \int \underbrace{(+\sin(x))}_{u} \underbrace{(+e^{-x}) dx}_{dv}$$

$$= -\cos(x) e^{-x} - \left(-\sin(x) e^{-x} - \int -e^{-x} \cos(x) \, dx \right)$$

$$\therefore \quad 2 \int \cos(x) e^{-x} dx = e^{-x} \left(\sin(x) - \cos(x) \right)$$