Mathematics 315 | Final Exam, Part II Spring 2006 | Name: Dr. Grow (1 pt.)

This portion of the 200 point final examination is "open book"; that is, you may freely use your two textbooks for this class: Rudin's *Principles of Mathematical Analysis* and Royden's *Real Analysis*. Work any three problems of your choosing, subject to the constraints that at least one problem must be chosen from Group A and at least one problem must be chosen from Group B. Please **CIRCLE** the numbers of the problems on this portion whose solutions you wish me to grade.

Group A.

4.(33 pts.) Let $\alpha(x)$ denote the fractional part of the real number x. For instance $\alpha(5/4)=.25$, $\alpha(2)=0$, and $\alpha(\pi)=.1415926...$

(a) Compute the total variation of α on the interval $[1,4]$.

(b) Show that the product of two functions of bounded variation on a closed bounded interval is of bounded variation on that interval.

(c) Let $f(x)=1/x$ and $\beta(x)=\alpha^2(x)$. Why is f Riemann-Stieltjes integrable with respect to β on the interval $[1,4]$?

(d) Evaluate the Riemann-Stieltjes integral $\int_1^4 f d\beta$.

5.(33 pts.) Let f be the 2π-periodic function defined on a fundamental period by the formula

$$f(x)=x^2-\frac{\pi^2}{3} \text{ if } -\pi\le x<\pi.$$

Show, by rigorous argument, that

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2t}$$

defines a function which solves the diffusion equation $u_t=u_{xx}$ in the region $t>0$ of the xt-plane and which satisfies the initial condition $u(x,0)=f(x)$ for $-\infty<x<\infty$.

6.(33 pts.) (a) If $k\in\mathbb{Z}$ and $f(x)=e^{ikx}$, show that

$$(*)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt.$$

(b) Show that (*) holds for every complex, continuous, 2π-periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, 2π-periodic function f on $\mathbb{R}$? Prove your assertion.

Group B.

7.(33 pts.) Let f be a function defined and bounded on the unit square

$$S=\{(x,t):\ 0<x<1,\ 0<t<1\}.$$

Suppose that:

(a) for each fixed t in $(0,1)$ the function $x\mapsto f(x,t)$ is measurable,

(b) at each (x,t) in S, the partial derivative $\frac{\partial f}{\partial t}$ exists, and

(c) $\frac{\partial f}{\partial t}$ is a bounded function in S.

Show that $\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)dx$.

8.(33 pts.) Let E denote the set of real numbers in the interval $[0,1]$ which possess a decimal expansion which contains no 2's and no 7's. For instance, the numbers $1/2 = .5$, and $7/10 = .6999\ldots$ belong to E, while the numbers $1/4 = .25 = .24999\ldots$ and $1/\sqrt{2} = .7071\ldots$ do not.

(a) Compute the Lebesgue measure of E.

(b) Determine, with proof, whether E is a Borel set.

9.(33 pts.) Let f be a function defined on the interval $[0,1]$ as follows: $f(x)=0$ if x is a point of the Cantor ternary set and $f(x)=1/k$ if x is in one of the complementary intervals of the Cantor set with length 3^{-k}. For example, $f(1/3)=0$, $f(1/2)=1$, and $f(4/5)=1/2$.

(a) Show that f is a measurable function.

(b) Evaluate the Lebesgue integral $\int_0^1 f(x)dx$.

(a) - (c) 8 pts. each (d) 9 pts.

#4 (a) Since α is right-continuous on [1,4] and α ↑ on $[1,2)$, $[2,3)$, and $[3,4)$, we have

$$\mathrm{Var}(\alpha;1,4) = \cancel{\alpha(2^-)}^{1} - \cancel{\alpha(1)}^{0} + \cancel{\alpha(2^-)}^{1} - \cancel{\alpha(2)}^{0} + \cancel{\alpha(3^-)}^{1} - \cancel{\alpha(2)}^{0} + \cancel{\alpha(3^-)}^{1} - \cancel{\alpha(3)}^{0}$$
$$+ \cancel{\alpha(4^-)}^{1} - \cancel{\alpha(3)}^{0} + \cancel{\alpha(4^-)}^{1} - \cancel{\alpha(4)}^{0}$$
$$= \boxed{6}.$$

(b) Let f and g be in $BV[a,b]$. For all $x \in [a,b]$,

$$|f(x)| \le |f(x) - f(a)| + |f(a)| \le \mathrm{Var}(f;a,x) + |f(a)| \le \mathrm{Var}(f;a,b) + |f(a)|,$$

so $M_f = \sup\{|f(x)| : x \in [a,b]\} \le \mathrm{Var}(f;a,b) + |f(a)| < \infty$.

Similarly, $M_g = \sup\{|g(x)| : x \in [a,b]\} < \infty$. Let $P = \{a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a,b]$. Then

$$\sum_{i=1}^{n} |\Delta(fg)_i| = \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|$$
$$\le \sum_{i=1}^{n} \Big(|f(x_i)g(x_i) - f(x_i)g(x_{i-1})| + |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-}$$
$$\le \sum_{i=1}^{n} M_f |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} M_g |f(x_i) - f(x_{i-1})|,$$

so taking the supremum over all partitions P of $[a,b]$ yields

$$\mathrm{Var}(fg;a,b) \le M_f \mathrm{Var}(g;a,b) + M_g \mathrm{Var}(f;a,b) < \infty.$$

(c) $f(x) = \frac{1}{x}$ is continuous on $[1,4]$ and $\beta = \alpha^2$ is of bounded variation on $[1,4]$ (see parts (a) & (b)). Consequently, the Riemann-Stieltjes integral $\int_1^4 f\,d\beta$ exists. (Cf. Theorem 6.8 in Rudin.)

(d) Using integration by parts and Theorem 6.17 in Rudin produces

$$\int_1^4 f\,d\beta = f(4)\cancelto{0}{\beta(4)} - f(1)\cancelto{0}{\beta(1)} - \int_1^4 \beta\,df = -\int_1^4 \alpha^2 f'\,dx = \int_1^4 \alpha^2(x)\frac{1}{x^2}\,dx$$

$$= \int_1^2 (x-1)^2\frac{1}{x^2}\,dx + \int_2^3 (x-2)^2\frac{1}{x^2}\,dx + \int_3^4 (x-3)^2\frac{1}{x^2}\,dx$$

$$= \int_1^2 \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx + \int_2^3 \left(1 - \frac{4}{x} + \frac{4}{x^2}\right)dx + \int_3^4 \left(1 - \frac{6}{x} + \frac{9}{x^2}\right)dx$$

$$= \left(x - 2\ln(x) - \frac{1}{x}\right)\Bigg|_1^2 + \left(x - 4\ln(x) - \frac{4}{x}\right)\Bigg|_2^3 + \left(x - 6\ln(x) - \frac{9}{x}\right)\Bigg|_3^4$$

$$= \boxed{\frac{59}{12} + 2\ln\left(\frac{3}{32}\right)} \doteq 0.1824$$

(#5) Let $\delta \in (0,1)$ and consider $H_\delta^+ = \{(x,t) \in \mathbb{R}^2 : t \geq \delta\}$. The sequence of functions $f_n(x,t) = \frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2 t}$ $(n = 1, 2, 3, \ldots,$ $(x,t) \in H_\delta^+)$ satisfies

$$\left|\frac{\partial f_n}{\partial t}(x,t)\right| = \left|4(-1)^{n+1}\cos(nx)e^{-n^2 t}\right| \leq 4e^{-n^2 t} \leq 4e^{-nt} \leq 4e^{-n\delta} \equiv M_n .$$

Since $\sum_{n=1}^{\infty} M_n = 4\sum_{n=1}^{\infty}(e^{-\delta})^n$ is a convergent geometric series, it follows from the Weierstrass M-test that $\sum_{n=1}^{\infty}\frac{\partial f_n}{\partial t}(x,t) = \sum_{n=1}^{\infty} 4(-1)^{n+1}\cos(nx)e^{-n^2}$ is uniformly convergent on H_δ^+. Clearly, for each fixed $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} f_n(x,1) = \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2}$ converges. (Use the Weierstrass M-test, for instance.) Therefore Theorem 7.17 in Rudin shows that

$$u(x,t) = \sum_{n=1}^{\infty} f_n(x,t) = \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2 t}$$

converges uniformly on H_δ^+ and

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to here

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\frac{\partial f_n}{\partial t}(x,t) = \sum_{n=1}^{\infty} 4(-1)^{n+1}\cos(nx)e^{-n^2 t}$$

for $(x,t) \in H_\delta^+$. Similarly

$$\left|\frac{\partial^2 f_n}{\partial x^2}(x,t)\right| = \left|4(-1)^{n+1}\cos(nx)e^{-n^2 t}\right| \leq 4e^{-\delta n} = M_n$$

$$\left|\frac{\partial f_n}{\partial x}(x,t)\right| = \left|\frac{4(-1)^{n+1}}{n}\sin(nx)e^{-n^2 t}\right| \leq \frac{4}{n}e^{-\delta n} = \frac{M_n}{n}$$

for $(x,t) \in H^+_\delta$ and $n = 1,2,3,\ldots$ with $\sum_{n=1}^{\infty} M_n$ and $\sum_{n=1}^{\infty} \frac{M_n}{n}$ convergent,

so by Theorem 7.17 in Rudin

$$\frac{\partial u}{\partial x}(x,t) = \sum_{n=1}^{\infty} \frac{\partial f_n}{\partial x}(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n} \sin(nx) e^{-n^2 t}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} \frac{\partial^2 f_n}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} 4(-1)^{n+1} \cos(nx) e^{-n^2 t}$$

14 pts. to here.

for $(x,t) \in H^+_\delta$. Note that $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ for all $(x,t) \in H^+_\delta$ so $u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) e^{-n^2 t}$ satisfies the diffusion equation $u_t = u_{xx}$ in H^+_δ. But $\delta \in (0,1)$ was arbitrary, so this function $u = u(x,t)$ satisfies the diffusion equation in $H^+ = \{(x,t) \in \mathbb{R}^2 : t > 0\}$.

16 to here.

It remains to show that

$$(*) \qquad u(x,0) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) = f(x) \qquad \text{for all real } x.$$

By routine calculations, we compute the Fourier coefficients of f:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} \left(x^2 - \frac{\pi^2}{3}\right) dx = \frac{1}{\pi}\left(\frac{x^3}{3} - \frac{\pi^2 x}{3}\right)\Big|_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{\underbrace{\left(x^2 - \frac{\pi^2}{3}\right)}_{\text{even}} \underbrace{\sin(nx)}_{\text{odd}}}^{\text{odd}} dx = 0 \qquad (n = 1,2,3,\ldots)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} \underbrace{\left(x^2 - \frac{\pi^2}{3}\right)}_{U} \underbrace{\cos(nx)\,dx}_{dV} = \cancel{\frac{2}{\pi}\left(x^2 - \frac{\pi^2}{3}\right) \sin(nx)\Big|_0^{\pi}} \xrightarrow{} 0$$

$$- \frac{2}{\pi} \int_0^{\pi} \frac{\sin(nx)}{n} 2x\,dx$$

$$= \frac{-4}{n\pi}\int_0^{\pi} \underbrace{x}_{U}\,\underbrace{\sin(nx)\,dx}_{dV} = \frac{-4}{n\pi}\left(\frac{-x\cos(nx)}{n}\Big|_0^{\pi} - \cancel{\int_0^{\pi} -\frac{\cos(nx)}{n}\,dx}^{\,0} \right)$$

21 pts. to here.

$$= \frac{4\cos(n\pi)}{n^2} = \frac{4(-1)^n}{n^2} \qquad (n = 1, 2, 3, \dots)$$

Therefore, the Fourier series of f is

24 pts. to here.

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx).$$

Note that $f'(x) = 2x$ if $-\pi < x < \pi$,

$$f_+'(-\pi) = \lim_{h\to 0^+} \frac{f(-\pi+h) - f(-\pi)}{h} = -2\pi,$$

and $$f_-'(-\pi) = \lim_{h\to 0^-} \frac{f(-\pi+h) - f(-\pi)}{h} = 2\pi.$$

Therefore, using 2π-periodicity of f and the Mean Value Theorem,

$$|f(x+t) - f(x)| \le 2\pi|t|$$

for all $x \in \mathbb{R}$ and all t sufficiently small in absolute value.

Theorem 8.14 in Rudin implies that f is equal to its Fourier series at each $x \in \mathbb{R}$; i.e. $u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2 t}$

33 pts. to here.

satisfies the boundary condition (*).

(#6) (a) If $k=0$ so $f(x)=e^{ikx}=1$ for all real x, then clearly $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$. Suppose k is a nonzero integer and $f(x)=e^{ikx}$. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{ikn}$$

A geometric series with first term e^{ik} and ratio $e^{ik} \neq 1$.

$$= \lim_{N\to\infty} \frac{1}{N}\left(\frac{e^{ik}-e^{ik(N+1)}}{1-e^{ik}}\right)$$

The numerator is a bounded function of N.

$$= 0.$$

On the other hand $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\,dt = \left.\frac{e^{ikt}}{2\pi i k}\right|_{t=-\pi}^{\pi}$

$$= \frac{e^{ik\pi}-e^{-ik\pi}}{2\pi i k} = 0.$$

Therefore, in every case when $f(x)=e^{ikx}$ for some integer k, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{=1}^{N} f(\;) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

(b) Let f be a complex, continuous, 2π-periodic function on $\mathbb{R}$ and let $\varepsilon>0$. By Theorem 8.15 in Rudin there exists a 2π-periodic trigonometric polynomial $P(x)=\sum_{k=-M}^{M} c_k e^{ikx}$ such that

$|P(x) - f(x)| < \varepsilon/3$ for all real x. By part (a), $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)d$

so there exists $N_0 \in \mathbb{N}$ such that $\left| \frac{1}{N}\sum_{n=1}^{N} P(n) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt \right| < \varepsilon/3$ for

all $N \geq N_0$. If $N \geq N_0$ then

$$\left| \frac{1}{N}\sum_{n=1}^{N} f(n) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt \right| \leq \frac{1}{N}\sum_{n=1}^{N} |f(n) - P(n)| + \left| \frac{1}{N}\sum_{n=1}^{N} P(n) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt \right|$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(t) - f(t)|dt$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Therefore $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$.

(c) No, (*) does not hold for every complex, bounded, measurable, 2π-periodic function on $\mathbb{R}$ for consider

$$f(t) = \begin{cases} 1 & \text{if } t - 2k\pi \in \mathbb{Z} \text{ for some integer } k, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly f is a 2π-periodic, complex (real, in fact!), bounded function on $\mathbb{R}$. Since there are only countably many points t in $\mathbb{R}$ such that $t - 2k\pi \in \mathbb{Z}$ for some integer k (they are all of the form $m + 2j\pi$ where $m, j \in \mathbb{Z}$), it follows that $f = 0$ a.e. so f is measurable.

However $f(n) = 1$ for all $n \in \mathbb{N}$ so

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = 1 \neq 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 0\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt .$$

(#7) Note that for each fixed $t \in (0,1)$, the function $x \mapsto f(x,t)$ is measurable and bounded, and hence belongs to $L^1(0,1)$. Let $F(t) = \int_0^1 f(x,t)\,dx$ for $t \in (0,1)$. In order to show that F is differentiable, we must show that for each $t_0 \in (0,1)$ and each sequence $\langle t_n \rangle_{n=1}^{\infty}$ such that $t_n \to t_0$ and $t_n \neq t_0$ for all $n \geq 1$, the limit $\lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0}$ exists and is independent of the sequence $\langle t_n \rangle$. To this end let t_0 and $\langle t_n \rangle$ be as above and define a sequence of functions on $(0,1)$ by

$$g_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \qquad (n = 1,2,3,\ldots)$$

By hypothesis $\lim_{n\to\infty} g_n(x) = \frac{\partial f}{\partial t}(x,t_0)$ for all $x \in (0,1)$.

The mean value theorem implies the existence of a number c_n between t_0 and t_n such that

$$f(x,t_n) - f(x,t_0) = \frac{\partial f}{\partial t}(x,c_n)\cdot(t_n - t_0),$$

so for all $x \in (0,1)$ and all $n \geq 1$, we have $|g_n(x)| \leq M$ where $M = \sup\left\{ \left|\frac{\partial f}{\partial t}(x,t)\right| : (x,t) \in S \right\} < \infty$ by hypothesis.

It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = \int_0^1 \frac{\partial f}{\partial t}(x, t_0)\,dx .$$

But

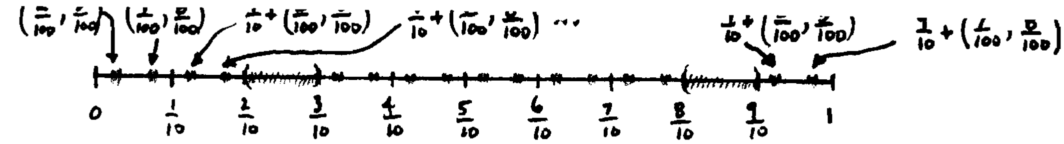
$$\lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n\to\infty} \frac{\int_0^1 f(x,t_n)\,dx - \int_0^1 f(x,t_0)\,dx}{t_n - t_0}$$

$$= \lim_{n\to\infty} \int_0^1 g_n(x)\,dx .$$

Therefore $F(t) = \int_0^1 f(x,t)\,dx$ is differentiable at each $t \in (0,1)$ with

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = F'(t) = \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx .$$

E_1 :



(#8) We imitate the construction of the Cantor ternary set to derive an alternate description of E. At the zeroth stage, we delete the two open intervals $\left(\frac{2}{10}, \frac{3}{10}\right)$ and $\left(\frac{7}{10}, \frac{8}{10}\right)$ from $[0,1]$ to obtain the closed set E_0. At the first stage, we delete the 16 open intervals

$$\frac{k}{10} + \left(\frac{2}{100}, \frac{3}{100}\right) \text{ and } \frac{k}{10} + \left(\frac{7}{100}, \frac{8}{100}\right) \qquad (k = 0,1,3,4,5,6,8,9)$$

from E_0 to obtain the closed set E_1. Continuing in this manner, at the k^{th} stage we delete $2 \cdot 8^k$ disjoint open intervals, each of length $10^{-(k+1)}$ from E_{k-1} to obtain the closed set E_k. It is clear that

$$E = \bigcap_{k=0}^{\infty} E_k .$$

16 pts. (b) E is closed, being the intersection of closed sets, so E is a Borel set.

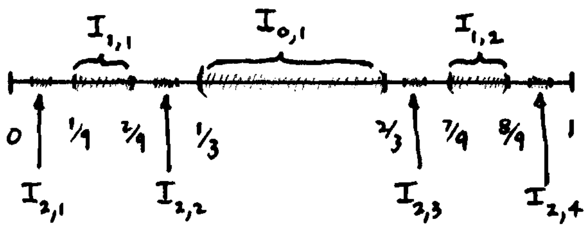
17 pts. (a) We compute the Lebesgue measure of E by first computing the measure of the open set $[0,1] \setminus E$. By construction $[0,1]\setminus E$ is countable union of disjoint open intervals so

$$m\left([0,1] \setminus E\right) = 2 \cdot \frac{1}{10} + 2 \cdot 8 \left(\frac{1}{100}\right) + 2 \cdot 8 \cdot 8 \left(\frac{1}{1000}\right) + \cdots$$

$$= \sum_{k=0}^{\infty} 2 \cdot 8^k \cdot 10^{-(k+1)} = \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^{k+1} = \frac{1}{4} \cdot \frac{4/5}{1 - 4/5} = 1 .$$

Therefore $\boxed{m(E) = 0}$.

(#9) Let $I_{k,j}$ $(j = 1, \ldots, 2^k)$ denote the j^{th} interval removed at the k^{th} step in the construction of the Cantor ternary set, arranged in ascending order; i.e. $I_{k,1} < I_{k,2} < \ldots < I_{k,2^k}$.



16 pts. (a) Note that $f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \frac{1}{k+1} \chi_{I_{k,j}}(x)$ for $0 \le x \le 1$,

so f is measurable, being the pointwise limit of a sequence of simple measurable functions: $f_K = \sum_{k=0}^{K} \sum_{j=1}^{2^k} \frac{1}{k+1} \chi_{I_{k,j}}$ $(K = 0, 1, 2, \ldots)$.

17 pts. (b)

$$\int_0^1 f\,dx \overset{\text{M.C.T.}}{=} \lim_{K\to\infty} \int_0^1 \Big(\sum_{k=0}^{K} \sum_{j=1}^{2^k} \frac{1}{k+1} \chi_{I_{k,j}}\Big) dx$$

$$= \lim_{K\to\infty} \sum_{k=0}^{K} \sum_{j=1}^{2^k} \frac{1}{k+1} m(I_{k,j})$$

$$= \lim_{K\to\infty} \sum_{k=0}^{K} \sum_{j=1}^{2^k} \frac{1}{k+1} \cdot 3^{-(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{2^k}{3^{k+1}}$$

(see next page for calculation)

$$= \frac{1}{2} \sum_{l=1}^{\infty} \frac{(2/3)^l}{l} = \boxed{\frac{1}{2} \ln(3)}$$

Computation for #9(b):

Since $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ if $|t| < 1$,

where the series converges uniformly on each compact subset of $(-1,1)$, we have for each $x \in (-1,1)$ that

(uniform convergence!)

$$\sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \int_0^x t^k dt = \int_0^x \Big(\sum_{k=0}^{\infty} t^k\Big) dt$$

$$= \int_0^x \frac{1}{1-t} dt = -\ln(1-t)\Big|_{t=0}^{x} = -\ln(1-x).$$

Therefore, setting $x = 2/3$ we have

$$\sum_{\ell=1}^{\infty} \frac{(2/3)^\ell}{\ell} = -\ln\left(1 - \tfrac{2}{3}\right) = \ln(3).$$