Exercises for Fourier Transform Methods for Solving PDE's

1.(a) Use Fourier transform methods to derive d'Alembert's solution to the initial value problem for the 1-D wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ -\infty < t < \infty,$$

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty.$$

(b) What assumptions on ϕ and ψ do you make in order for the derivation in part (a) to be rigorous?

2. Solve problem # 16 in Sec. 2.4 by Fourier transform methods.

3. Solve problem # 17 in Sec. 2.4 by Fourier transform methods.

4. Solve problem # 18 in Sec. 2.4 by Fourier transform methods.

5. Let f be a piecewise-continuous absolutely integrable function on $-\infty < x < \infty$.

(a) Use Fourier transform methods to solve the 2-D Laplace equation

$$u_{xx} + u_{yy} = 0 \quad \text{in the upper halfplane } -\infty < x < \infty,\ 0 < y < \infty$$

subject to the boundary condition

$$u(x,0) = f(x) \quad \text{for } -\infty < x < \infty$$

and the decay condition

$$u(x,y) \longrightarrow 0 \quad \text{as} \quad x^2 + y^2 \longrightarrow \infty.$$

(b) Let $f(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$ Compute an explicit formula for the solution $u = u(x,y)$ in part (a).

Exercises for Fourier Transform Methods

1. (a) Use Fourier transform methods to derive d'Alembert's solution to the initial value problem for the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ -\infty < t < \infty,$$

$$u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty.$$

(b) What assumptions on φ and ψ do you make in order for the derivation in part (a) to be rigorous?

We will make use of the following result in part (a).

FACT: Let f be a piecewise-continuous absolutely integrable function on $(-\infty,\infty)$ such that $\hat{f}(0)=0$, and let

$$F(x) = \int_{-\infty}^{x} f(y)\,dy, \qquad x \in (-\infty,\infty).$$

Then $\hat{F}(\xi) = \dfrac{\hat{f}(\xi)}{i\xi}$ for $\xi \neq 0$.

Proof of FACT: If $\xi \neq 0$ then

$$\hat{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-i\xi x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{x} f(y)\,dy \right)}_{U} \underbrace{e^{-i\xi x}\,dx}_{dV} \qquad \text{(Integrate by parts.)}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\int_{-\infty}^{x} f(y)\,dy \right) \left(\frac{e^{-i\xi x}}{-i\xi} \right) \Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{-i\xi} f(x)\,dx \right]$$

But $\displaystyle\lim_{x\to-\infty} \int_{-\infty}^{x} f(y)\,dy = 0$, $\displaystyle\lim_{x\to\infty} \int_{-\infty}^{x} f(y)\,dy = \sqrt{2\pi}\,\hat{f}(0) = 0$,

and $\left| \dfrac{e^{-i\xi x}}{-i\xi} \right| = \dfrac{1}{\xi}$ for all real x. It follows that

Exercises for Fourier Transform Methods (cont.)

1. (cont.) $\hat{F}(\xi) = \frac{1}{i\xi}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx = \frac{\hat{f}(\xi)}{i\xi}$.

(a) $\mathcal{F}(u_{tt} - c^2u_{xx}) = \mathcal{F}(0)$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u) + c^2\xi^2\mathcal{F}(u) = 0$$

$$\mathcal{F}(u) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t)$$

$$\hat{\varphi}(\xi) = \mathcal{F}(u(\cdot,0)) = c_1(\xi)$$

$$\hat{\psi}(\xi) = \mathcal{F}(u_t(\cdot,0)) = \frac{\partial}{\partial t}\mathcal{F}(u)\Big|_{t=0} = -c\xi c_1(\xi)\sin(c\xi t) + c\xi c_2(\xi)\cos(c\xi t)\Big|_{t=0}$$

$$\hat{\psi}(\xi) = c\xi c_2(\xi)$$

$$\therefore\quad \mathcal{F}(u) = \hat{\varphi}(\xi)\cos(c\xi t) + \frac{\hat{\psi}(\xi)}{c\xi}\sin(c\xi t)$$

$$= \tfrac{1}{2}\hat{\varphi}(\xi)e^{ic\xi t} + \tfrac{1}{2}\hat{\varphi}(\xi)e^{-ic\xi t} + \frac{\hat{\psi}(\xi)}{c\xi}\left(\frac{e^{ic\xi t} - e^{-ic\xi t}}{2i}\right)$$

$$= \tfrac{1}{2}\hat{\varphi}(\xi)e^{ic\xi t} + \tfrac{1}{2}\hat{\varphi}(\xi)e^{-ic\xi t} + \frac{1}{2c}\frac{\hat{\psi}(\xi)}{i\xi}e^{ic\xi t} - \frac{1}{2c}\frac{\hat{\psi}(\xi)}{i\xi}e^{-ic\xi t}.$$

If $\Psi(x) = \int_{-\infty}^{x}\psi(y)\,dy$ then $\hat{\Psi}(\xi) = \frac{\hat{\psi}(\xi)}{i\xi}$ by FACT. Thus

$$(*)\qquad \mathcal{F}(u) = \tfrac{1}{2}\hat{\varphi}(\xi)e^{ic\xi t} + \tfrac{1}{2}\hat{\varphi}(\xi)e^{-ic\xi t} + \frac{1}{2c}\hat{\Psi}(\xi)e^{ic\xi t} - \frac{1}{2c}\hat{\Psi}(\xi)e^{-ic\xi t}.$$

Fix the time t; by the "shifting on the x-axis" result (#4 on the Exercises for Fourier Transforms),

$f_1(x) = \varphi(x+ct)$ has Fourier transform $\hat{f}_1(\xi) = e^{i\xi ct}\hat{\varphi}(\xi)$;

$f_2(x) = \varphi(x-ct)$ " " " $\hat{f}_2(\xi) = e^{-i\xi ct}\hat{\varphi}(\xi)$;

$g_1(x) = \Psi(x+ct)$ has Fourier transform $\hat{g}_1(\xi) = e^{i\xi ct}\hat{\Psi}(\xi)$;

$g_2(x) = \Psi(x-ct)$ " " " $\hat{g}_2(\xi) = e^{-i\xi ct}\hat{\Psi}(\xi)$.

Substituting these relations into (*) gives

$$\mathcal{F}(u) = \mathcal{F}\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2 + \tfrac{1}{2c}g_1 - \tfrac{1}{2c}g_2\right),$$

and the uniqueness theorem implies (for fixed t and any real x)

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}f_1(x) + \tfrac{1}{2}f_2(x) + \tfrac{1}{2c}g_1(x) - \tfrac{1}{2c}g_2(x) \\ &= \tfrac{1}{2}\varphi(x+ct) + \tfrac{1}{2}\varphi(x-ct) + \tfrac{1}{2c}\Psi(x+ct) - \tfrac{1}{2c}\Psi(x-ct) \\ &= \tfrac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \tfrac{1}{2c}\left[\int_{-\infty}^{x+ct}\psi(y)\,dy - \int_{-\infty}^{x-ct}\psi(y)\,dy\right]. \end{aligned}$$

$$\therefore \quad \boxed{u(x,t) = \tfrac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy}$$

(b) In order for the function $u = u(x,t)$ above to satisfy the P.D.E., it is clear that we must have $\boxed{\varphi \in C^2(-\infty,\infty) \text{ and } \psi \in C^1(-\infty,\infty).}$ In the derivation by Fourier transform methods in part (a), we applied FACT with $f = \psi$. Therefore $\boxed{\psi \text{ must be absolutely integrable on } (-\infty,\infty)}$. Since we take the Fourier transform of φ, it is natural to require that $\boxed{\varphi \text{ be absolutely integrable on } (-\infty,\infty)}$. Finally, we interchange integration and differentiation when we write $\frac{\partial^2}{\partial t^2}\mathcal{F}(u) = \mathcal{F}(u_{tt})$. Thus, by

Exercises for Fourier Transform Methods (cont.)

1 (b) (cont.) Theorem 2 of A.3 (see p. 390), it is natural to require that
$\boxed{\varphi'' \text{ and } \psi' \text{ be absolutely integrable on } (-\infty, \infty).}$

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2. Solve problem #16 in Sec. 2.4 by Fourier transform methods.
" Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

with $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$. Here $b > 0$ is constant."

Taking the Fourier transform of both sides of the PDE with respect to the variable x yields

$$\mathcal{F}(u_t - ku_{xx} + bu) = \mathcal{F}(0)$$

$$\frac{\partial}{\partial t}\mathcal{F}(u) - k(-\xi^2)\mathcal{F}(u) + b\mathcal{F}(u) = 0.$$

$$\therefore \quad \mathcal{F}(u) = c(\xi)e^{-(k\xi^2+b)t}.$$

$$\mathcal{F}(\varphi) = \mathcal{F}(u(\cdot,0)) = c(\xi)e^{0} = c(\xi),$$

$$(*) \qquad \therefore \quad \mathcal{F}(u) = \mathcal{F}(\varphi)e^{-(k\xi^2+b)t}$$

Using formula I in the table of Fourier transforms with $kt = \frac{1}{4a}$ yields

$$\frac{1}{\sqrt{2kt}}\mathcal{F}\left(e^{\frac{-(\cdot)^2}{4kt}}\right)e^{-bt} = \frac{1}{\sqrt{2kt}}\cdot\sqrt{2kt}\ e^{-kt\xi^2}\cdot e^{-bt}$$

$$= e^{-(k\xi^2+b)t}.$$

Substituting this expression into (*) produces

2. (cont.)
$$\mathcal{F}(u) = \mathcal{F}(\varphi)\frac{1}{\sqrt{2kt}}\mathcal{F}\left(e^{-\frac{(\cdot)^2}{4kt}}\right)e^{-bt}$$
$$= \frac{e^{-bt}}{\sqrt{2kt}}\cdot\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right)$$
$$= \mathcal{F}\left(\frac{e^{-bt}}{\sqrt{4k\pi t}}\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right)$$

By the uniqueness theorem (for fixed $t>0$ and any real x) it follows that
$$u(x,t) = \frac{e^{-bt}}{\sqrt{4k\pi t}}\left(\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right)(x),$$
i.e.
$$\boxed{u(x,t) = \frac{e^{-bt}}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy}\,.$$

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3. Solve problem #17 in Sec. 2.4 by Fourier transform methods.
" Solve the diffusion equation with variable dissipation:
$$u_t - ku_{xx} + bt^2u = 0$$
for $-\infty<x<\infty$, $0<t<\infty$, with $u(x,0)=\varphi(x)$ for $-\infty<x<\infty$; here $b>0$ is a constant. "

Taking the Fourier transform (with respect to x) of both sides of the PDE yields
$$\mathcal{F}(u_t) - k\mathcal{F}(u_{xx}) + bt^2\mathcal{F}(u) = 0$$
$$\frac{\partial}{\partial t}\mathcal{F}(u) - k(i\xi)^2\mathcal{F}(u) + bt^2\mathcal{F}(u) = 0$$

3. (cont.) $$\frac{\partial}{\partial t}\mathcal{F}(u) + (k\xi^2 + bt^2)\mathcal{F}(u) = 0.$$

Separating variables and integrating produces

$$\ln \mathcal{F}(u) = -k\xi^2 t - \frac{bt^3}{3} + c(\xi)$$

or $$\mathcal{F}(u) = A(\xi)e^{-k\xi^2 t - \frac{bt^3}{3}}. \qquad \left(\text{where } A(\xi) = e^{c(\xi)}.\right)$$

Applying the initial condition we have

$$\mathcal{F}(\varphi) = \mathcal{F}(u(\cdot,0)) = A(\xi)e^0 = A(\xi)$$

so

$$(\dagger) \quad \mathcal{F}(u) = \mathcal{F}(\varphi)\, e^{-kt\xi^2} \cdot e^{-\frac{bt^3}{3}}.$$

Applying formula I: $\mathcal{F}\left(e^{-a(\cdot)^2}\right) = \dfrac{e^{-\frac{\xi^2}{4a}}}{\sqrt{2a}}$, with $kt = \dfrac{1}{4a}$ (that is, $a = \dfrac{1}{4kt}$) gives $\mathcal{F}\left(e^{-\frac{(\cdot)^2}{4kt}}\right) = \sqrt{2kt}\, e^{-kt\xi^2}$. Substituting this expression into $(\dagger)$, we find

$$\mathcal{F}(u) = \mathcal{F}(\varphi)\mathcal{F}\left(e^{-\frac{(\cdot)^2}{4kt}}\right) \cdot \frac{e^{-\frac{bt^3}{3}}}{\sqrt{2kt}}.$$

Using the convolution formula $\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$ we have

$$\mathcal{F}(u) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right) \cdot \frac{e^{-\frac{bt^3}{3}}}{\sqrt{2kt}}$$

$$= \mathcal{F}\left(\frac{e^{-\frac{bt^3}{3}}}{\sqrt{4k\pi t}}\, \varphi * e^{-\frac{(\cdot)^2}{4kt}}\right).$$

By the uniqueness theorem (for fixed $t > 0$ and any real x)

$$u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4k\pi t}}\left(\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right)(x)$$

i.e. $$\boxed{u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy.}$$

Exercises for Fourier Transform Methods (cont.)

4. Solve problem #18 in Sec. 2.4 by Fourier transform methods.
" Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

with $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$, where V is a constant."

Taking the Fourier transform (with respect to x) of the PDE yields

$$\mathcal{F}(u_t) - k\mathcal{F}(u_{xx}) + V\mathcal{F}(u_x) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u) - k(i\xi)^2\mathcal{F}(u) + V(i\xi)\mathcal{F}(u) = 0$$

$$(\dagger) \qquad \frac{\partial}{\partial t}\mathcal{F}(u) + (k\xi^2 + iV\xi)\mathcal{F}(u) = 0.$$

An integrating factor for this linear first-order equation in t (for fixed ξ) is

$$e^{\int (k\xi^2 + iV\xi)\,dt} = e^{(k\xi^2+iV\xi)t}.$$

Multiplying $(\dagger)$ by the integrating factor and using the product rule for derivatives gives

$$\frac{\partial}{\partial t}\left\{ \mathcal{F}(u)\, e^{(k\xi^2+iV\xi)t} \right\} = 0,$$

whereupon integration and algebra yields

$$\mathcal{F}(u) = c(\xi)e^{-(k\xi^2+iV\xi)t}.$$

Applying the initial condition, we have

$$\mathcal{F}(\varphi) = \mathcal{F}(u(\cdot,0)) = c(\xi)e^0 = c(\xi).$$

Thus $(*)$ $\quad \mathcal{F}(u) = \mathcal{F}(\varphi)\cdot e^{-kt\xi^2} \cdot e^{-iVt\xi}.$

Exercises for Fourier Transform Methods (cont.)

4. (cont.) As in problems 2 and 3, $\mathcal{F}\left(e^{-\frac{(\cdot)^2}{4kt}}\right) = \sqrt{2kt}\, e^{-kt\xi^2}$, so substituting in (*) we have

$$\mathcal{F}(u) = \mathcal{F}(\varphi)\mathcal{F}\left(\frac{1}{\sqrt{2kt}} e^{-\frac{(\cdot)^2}{4kt}}\right) e^{-iVt\xi},$$

and using the convolution formula as in problems 2 and 3,

$$(**) \qquad \mathcal{F}(u) = e^{-iVt\xi}\,\mathcal{F}\left(\frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} * \varphi\right).$$

Applying the shifting formula on the x-axis (#4 on Exercises for Fourier Transforms): $\mathcal{F}(f(\cdot - a)) = e^{-i\xi a}\hat{f}(\xi)$, with $a = Vt$, (**) becomes

$$\mathcal{F}(u) = \mathcal{F}\left(\frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot - Vt)^2}{4kt}} * \varphi\right).$$

The uniqueness theorem then implies (for fixed $t > 0$ and any real x)

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}}\left(e^{-\frac{(\cdot - Vt)^2}{4kt}} * \varphi\right)(x)$$

i.e. $$\boxed{u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y-Vt)^2}{4kt}}\varphi(y)\,dy}.$$

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5. Let f be an absolutely integrable, piecewise-continuous function on $-\infty < x < \infty$.

(a) Use Fourier transform methods to solve the 2-D Laplace equation $u_{xx} + u_{yy} = 0$ in the upper halfplane $-\infty < x < \infty$, $0 < y < \infty$, subject to the boundary condition $u(x,0) = f(x)$ for $-\infty < x < \infty$ and

5. (cont.) the decay condition $u(x,y) \to 0$ as $x^2+y^2 \to \infty$.

(b) Let
$$f(x) = \begin{cases} 1 & \text{if } |x|<1, \\ 0 & \text{otherwise}. \end{cases}$$

Compute an explicit formula for the solution $u = u(x,y)$ in part (a).

(a) We take the Fourier transform (with respect to x) of the PDE:

$$\mathcal{F}(u_{xx}) + \mathcal{F}(u_{yy}) = 0$$

$$-\xi^2 \mathcal{F}(u) + \frac{\partial^2}{\partial y^2}\mathcal{F}(u) = 0.$$

$$(\dagger) \qquad \mathcal{F}(u) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}.$$

Applying the boundary condition yields

$$(\dagger\dagger) \qquad \mathcal{F}(f)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c_1(\xi)e^{0} + c_2(\xi)e^{0} = c_1(\xi) + c_2(\xi)$$

for $-\infty < \xi < \infty$. The decay condition implies

$$(*) \qquad 0 = \lim_{|y|\to\infty} \mathcal{F}(u) = \lim_{|y|\to\infty}\left(c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}\right)$$

for $-\infty < \xi < \infty$.

Suppose $\xi > 0$; let $y \to +\infty$ in $(*)$ and observe that we must have $c_1(\xi) = 0$ if $(*)$ is to be satisfied. Suppose $\xi < 0$; let $y \to +\infty$ in $(*)$ and observe that this time we must have $c_2(\xi) = 0$. Using these relations in conjunction with $(\dagger\dagger)$ yields

$$(\dagger\dagger\dagger) \qquad c_1(\xi) = \begin{cases} \mathcal{F}(f)(\xi) & \text{if } \xi < 0, \\ 0 & \text{if } \xi > 0, \end{cases} \quad \text{and} \quad c_2(\xi) = \begin{cases} 0 & \text{if } \xi < 0, \\ \mathcal{F}(f)(\xi) & \text{if } \xi > 0. \end{cases}$$

5. (cont.) By (†) and (†††),

$$\mathcal{F}(u) = \begin{cases} \mathcal{F}(f)(\xi)\, e^{\xi y} & \text{if } \xi < 0, \\ \mathcal{F}(f)(\xi)\, e^{-\xi y} & \text{if } \xi > 0, \end{cases}$$

or equivalently,

$$(\dagger\dagger\dagger\dagger) \quad \mathcal{F}(u) = \mathcal{F}(f)(\xi)\, e^{-|\xi| y}.$$

Applying formula C for Fourier transforms:

$$\mathcal{F}\left(\frac{1}{(\cdot)^2 + a^2}\right) = \sqrt{\frac{\pi}{2}}\; \frac{e^{-a|\xi|}}{a}$$

with $a = y > 0$, we find that $\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\, \frac{y}{(\cdot)^2 + y^2}\right) = e^{-y|\xi|}$.

Substituting in (††††) and using the convolution formula $\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\mathcal{F}(g)$, we have

$$\mathcal{F}(u) = \mathcal{F}(f)\, \mathcal{F}\left(\sqrt{\frac{2}{\pi}} \cdot \frac{y}{(\cdot)^2 + y^2}\right) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left(f * \sqrt{\frac{2}{\pi}}\, \frac{y}{(\cdot)^2 + y^2}\right),$$

whereupon the uniqueness theorem (for fixed $y > 0$ and all real x) gives

$$u(x,y) = \frac{1}{\pi}\left(f * \frac{y}{(\cdot)^2 + y^2}\right)(x)$$

i.e.

$$\boxed{u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, f(s)\, ds}{(x-s)^2 + y^2}}.$$

(b) If $f(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise}, \end{cases}$ then the solution to the problem in part (a) is

5. (cont.) $$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(s)\,ds}{(x-s)^2+y^2} = \frac{1}{\pi}\int_{-1}^{1} \frac{y \cdot 1\,ds}{(x-s)^2+y^2}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{\frac{1}{y}\,ds}{\left(\frac{x-s}{y}\right)^2+1}.$$ Let $p = \frac{s-x}{y}$. Then $dp = \frac{1}{y}ds$ so

$$u(x,y) = \frac{1}{\pi}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}} \frac{dp}{p^2+1} = \frac{1}{\pi}\left[\text{Arctan}\left(\frac{1-x}{y}\right) - \text{Arctan}\left(\frac{-1-x}{y}\right)\right].$$

$$\therefore \quad \boxed{u(x,y) = \frac{1}{\pi}\left[\text{Arctan}\left(\frac{1-x}{y}\right) + \text{Arctan}\left(\frac{1+x}{y}\right)\right].}$$