Mathematics 325
Homework 9

Due Date: ____________________

Name: ________________________

Work exercise 2 on page 50 in Strauss.

1. Solve the diffusion equation with the initial condition
$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$
Write your answer in terms of $\mathcal{E}\mathrm{rf}(x)$.
2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

$$\begin{cases} u_t - k u_{xx} = 0 & \text{in } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \varphi(x) = \begin{cases} 1 & \text{if } x > 0, \\ 3 & \text{if } x < 0. \end{cases} \end{cases}$$

$$\therefore\ u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{0} 3e^{-\frac{(x-y)^2}{4kt}}\,dy + \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} 1e^{-\frac{(x-y)^2}{4kt}}\,dy$$

Let $p = \frac{y-x}{\sqrt{4kt}}$. Then $dp = \frac{dy}{\sqrt{4kt}}$.

$$\therefore\ u(x,t) = \int_{-\infty}^{-\frac{x}{\sqrt{4kt}}} \frac{3}{\sqrt{\pi}} e^{-p^2}\,dp + \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\,dp$$

Note that $\mathrm{Erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-p^2}\,dp = \frac{2}{\sqrt{\pi}}\left(\int_{-\infty}^{w} e^{-p^2}\,dp - \int_{-\infty}^{0} e^{-p^2}\,dp \right) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{w} e^{-p^2}\,dp - 1$.

Therefore $\int_{-\infty}^{w} e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}\left(1 + \mathrm{Erf}(w)\right)$. A similar argument shows that

$\int_{w}^{\infty} e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{Erf}(w)\right)$. Therefore

$$u(x,t) = \frac{3}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\left(1 + \mathrm{Erf}\left(\frac{-x}{\sqrt{4kt}}\right)\right) + \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{Erf}\left(\frac{-x}{\sqrt{4kt}}\right)\right)$$

$$= 2 + \mathrm{Erf}\left(\frac{-x}{\sqrt{4kt}}\right)$$

$$= \boxed{2 - \mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right).}$$