Mathematics 325
Homework 13

Due Date: ____________________

Name: __________________________

Solve exercise #2 on page 87 of Strauss.

2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$

HW 13

Sec 4.1, #2 Due: 7/10

25% $$\begin{cases} u_t - ku_{xx} \overset{①}{=} 0 & \text{for } 0<x<\ell,\ 0<t<\infty, \\ u(0,t) \overset{②}{=} 0 \overset{③}{=} u(\ell,t) & \text{for } 0<t<\infty, \\ u(x,0) \overset{④}{=} 1 & \text{for } 0<x<\ell \end{cases}$$

$u(x,t) = X(x)T(t)$ in ①-②-③ leads to

40% $$X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(\ell)$$
$$T'(t) + \lambda k T(t) = 0$$

The eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ and eigenfunctions $X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$ $(n=1,2,3,\ldots)$

55% The solutions to the T-equation are $T_n(t) = e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$ $(n=1,2,3,\ldots)$ Consequently

% $$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \qquad (b_1, b_2, b_3, \ldots \text{ arbitrary constants})$$

is a formal solution to ①-②-③. We want to chose the coefficients so ④ is satisfied. Using the assumed identity for the sine series expansion of 1 this means

$$\frac{4}{\pi}\left(\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{\ell}\right) + \ldots\right) = 1 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

for all $0<x<\ell$. By inspection, we see that

85% $$\frac{4}{\pi} = b_1, \quad \frac{4}{3\pi} = b_3, \quad \frac{4}{5\pi} = b_5, \quad \ldots \qquad \text{and } 0 = b_2 = b_4 = \ldots$$

That is, $b_{2n-1} = \frac{4}{\pi(2n-1)}$ and $b_{2n} = 0$ for $n=1,2,3,\ldots$ Hence, the solution to ①-②-③-④ is

100% $$\boxed{u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/\ell)\, e^{-\frac{(2n-1)^2\pi^2}{\ell^2}kt}}{2n-1}}.$$