Mathematics 315 | Midterm Exam Spring 2007 | Name: Dr. Grow

1.(25 pts.) Let f be a bounded real function on $[a,b]$ and let α be a real function on $[a,b]$.

(a) If α is increasing on $[a,b]$, define the phrase "f is Riemann-Stieltjes integrable with respect to α on $[a,b]$".

(b) Define what it means for α to be of bounded variation on $[a,b]$.

(c) Give an example of a function which is differentiable but not of bounded variation on $[a,b]$.

(d) State a condition on the derivative of a differentiable function which will guarantee that the function is of bounded variation on $[a,b]$.

(e) State Jordan's theorem relating functions of bounded variation and increasing functions.

(f) State the definition of the Riemann-Stieltjes integral of f with respect to α on $[a,b]$ if f is continuous on $[a,b]$ and α is of bounded variation on $[a,b]$.

(g) If f is continuous on $[0,1]$ and $\alpha(x)=\sum_{n=2}^{\infty}\frac{1}{n^2}H\left(x-\frac{1}{n}\right)$, write, without proof, a formula for the value of $\int_0^1 f d\alpha$. (Here H denotes the unit Heaviside step function.)

(h) If f is Riemann integrable on $[0,1]$ and α is differentiable with α' Riemann integrable on $[0,1]$, write, without proof, a formula for the value of $\int_0^1 f d\alpha$.

2.(25 pts.) Consider the vector space $C[a,b]$ of all continuous real functions on the interval $[a,b]$.

(a) Define the phrase "N is a norm on $C[a,b]$".

(b) If N is a norm on $C[a,b]$, define the phrase "N is a Banach space norm on $C[a,b]$".

(c) Give, without proof, an example of a norm on $C[a,b]$ which is **not** a Banach space norm.

(d) Give, without proof, an example of a norm on $C[a,b]$ which **is** a Banach space norm.

(e) Define the phrase "Λ is a bounded linear functional on the normed linear space $(C[a,b],N)$".

(f) State, without proof, the Riesz Representation Theorem characterizing the bounded linear functionals on the space $C[a,b]$. (Be sure to explicitly state the Banach space norm that is being used on $C[a,b]$.)

(g) Show that $\Lambda(f)=3f(1/2)-\int_0^1\frac{f(x)}{\sqrt{x}}dx$ defines a bounded linear functional on $C[0,1]$ equipped with an appropriate Banach space norm. Then find a function α corresponding to Λ guaranteed by the Riesz Representation Theorem.

3.(25 pts.) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions on $[a,b]$, and let f be a real function on $[a,b]$.

(a) Define the phrase "$\{f_n\}_{n=1}^{\infty}$ converges to f pointwise on $[a,b]$".

(b) Define the phrase "$\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on $[a,b]$".

(c) Give, without proof, an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ which is pointwise convergent but not uniformly convergent on $[0,1]$.

(d) State the Stone-Weierstrass approximation theorem.

(e) Let $\mathcal{B}$ denote the family of all real functions on the closed bounded rectangle $\mathcal{R} = \{(x, y) : x \in [a,b], y \in [c,d]\}$ of the form

$$F(x,y) = \sum_{k=1}^{n} f_k(x) g_k(y)$$

where n is a positive integer, each $f_k \in C[a,b]$, and each $g_k \in C[c,d]$. Briefly sketch the steps you would take in showing that to any $f \in C(\mathcal{R})$ and any $\varepsilon > 0$ there corresponds $F \in \mathcal{B}$ such that $|f(x,y) - F(x,y)| < \varepsilon$ for all $(x,y) \in \mathcal{R}$.

4.(25 pts.) Let $\{f_n\}_{n=1}^{\infty}$ be a pointwise bounded sequence of complex functions on a countable set E. Show that $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges for every x in E.

#1 (a) Let $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ where $P: a=x_0<x_1<\dots<x_n=b$ is a partition of $[a,b]$ and $M_i = \sup\{f(x): x\in[x_{i-1},x_i]\}$ for $1\le i\le n$. Let $L(P,f,\alpha) = \sum_{i=1}^{n} m_i\Delta\alpha_i$ where $m_i = \inf\{f(x): x\in[x_{i-1},x_i]\}$ $(1\le i\le n)$. Define

$\overline{\int_a^b} f\,d\alpha = \inf\{U(P,f,\alpha): P \text{ is a partition of } [a,b]\}$ and $\underline{\int_a^b} f\,d\alpha = \sup\{L(P,f,\alpha): P$ is a partition of $[a,b]\}$. We say that "f is Riemann-Stieltjes integrable with respect to

+3 α on $[a,b]$" provided $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$.

+3 (b) Define $\operatorname{Var}(\alpha; a,b) = \sup\left\{\sum_{i=1}^{n}|\alpha(x_i)-\alpha(x_{i-1})| : P=\{a=x_0,x_1,\dots,x_n=b\} \text{ is a partition of } [a,b]\right\}$. We say "$\alpha$ is of bounded variation on $[a,b]$" provided $\operatorname{Var}(\alpha;a,b)<\infty$

+3 (c) $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x^2}\right) & \text{if } x\neq 0, \\ 0 & \text{if } x=0, \end{cases}$ is differentiable on $[0,1]$ but not of bounded variation on $[0,1]$.

+3 (d) If $|f'(x)|\le M<\infty$ for all $x\in[a,b]$ then $f\in BV[a,b]$.

+4 (e) Let f be of bounded variation on $[a,b]$. Then there exist increasing real-valued functions f_1 and f_2 on $[a,b]$ such that $f=f_1-f_2$.

+3 (f) $\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 - \int_a^b f\,d\alpha_2$ where α_1 and α_2 are increasing real-valued functions on $[a,b]$ such that $\alpha=\alpha_1-\alpha_2$.

+3 (g) $\int_a^b f\,d\alpha = \sum_{n=2}^{\infty}\frac{1}{n^2}f\left(\frac{1}{n}\right)$.

+3 (h) $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$.

#2 (a) "N is a norm on $C[a,b]$" means that N is a real-valued function defined on $C[a,b]$ with the following properties:

+3

(i) $N(f) \geq 0$ for all $f \in C[a,b]$, with equality only if $f = 0$;

(ii) $N(cf) = |c| N(f)$ for all $c \in \mathbb{R}$ and all $f \in C[a,b]$;

(iii) $N(f+g) \leq N(f) + N(g)$ for all f and g in $C[a,b]$.

+3 (b) "N is a Banach space norm on $C[a,b]$" provided N is a norm on $C[a,b]$ in which every Cauchy sequence $\{f_n\}$ is convergent. That is, if $\{f_n\}$ is any sequence in $C[a,b]$ with the property that to each $\varepsilon > 0$ there corresponds an integer $M \geq 1$ such that $N(f_n - f_m) < \varepsilon$ for all $m, n \geq M$, then there corresponds a function $f \in C[a,b]$ such that $N(f - f_n) \to 0$ as $n \to \infty$.

+3 (c) $N_2(f) = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$.

+3 (d) $N(f) = \sup\{ |f(x)| : x \in [a,b] \}$ $(= \|f\|_u$, the uniform norm of f.)

+3 (e) "Λ is a bounded linear functional on the normed linear space $(C[a,b], N)$" means that Λ is a real-valued function defined on $C[a,b]$ with the following properties:

(i) $\Lambda(c_1 f_1 + c_2 f_2) = c_1 \Lambda(f_1) + c_2 \Lambda(f_2)$ for all $c_1, c_2 \in \mathbb{R}$ and all $f_1, f_2 \in C[a,b]$;

(ii) there exists a real number K such that $|\Lambda(f)| \leq K N(f)$ for all $f \in C[a,b]$.

+4 (f) If Λ is a bounded linear functional on $(C[a,b], \|\cdot\|_u)$ then there is a function α of bounded variation on $[a,b]$ such that $\Lambda(f) = \int_a^b f \, d\alpha$ for all $f \in C[a,b]$.

(g) If $f \in C[0,1]$ then $\int_0^1 \left| \frac{f(x)}{\sqrt{x}} \right| dx \leq \int_0^1 \frac{\|f\|_u}{\sqrt{x}} dx = 2\|f\|_u$ so the improper integral $\int_0^1 \frac{f(x)}{\sqrt{x}} dx$ is absolutely convergent and hence Λ defines a real-valued function on $C[0,1]$. Suppose $c_1, c_2 \in \mathbb{R}$ and $f_1, f_2 \in C[0,1]$. Then

$$\Lambda(c_1 f_1 + c_2 f_2) = 3(c_1 f_1 + c_2 f_2)(\tfrac{1}{2}) - \int_0^1 \frac{c_1 f_1(x) + c_2 f_2(x)}{\sqrt{x}} dx = 3c_1 f_1(\tfrac{1}{2}) - c_1 \int_0^1 \frac{f_1(x)}{\sqrt{x}} dx + 3c_2 f_2(\tfrac{1}{2}) - c_2 \int_0^1 \frac{f_2(x)}{\sqrt{x}} dx$$

$= c_1\Lambda(f_1) + c_2\Lambda(f_2)$, so Λ is linear.

Suppose $f \in C[0,1]$. Then

$$|\Lambda(f)| = \left| 3f(1/2) - \int_0^1 \frac{f(x)}{\sqrt{x}}dx \right| \le 3|f(1/2)| + \int_0^1 \frac{|f(x)|}{\sqrt{x}}dx \le 3\|f\|_u + \int_0^1 \frac{\|f\|_u}{\sqrt{x}}dx$$

+3 here $$\le 3\|f\|_u + 2\|f\|_u = 5\|f\|_u .$$

Therefore Λ is a bounded linear functional on $(C[0,1], \|\cdot\|_u)$. (Oops! Forgot α. OVER.)

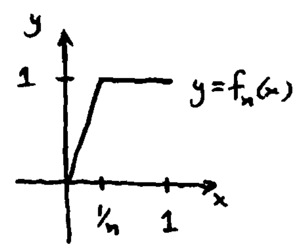
#3. (a) "$\{f_n\}$ converges to f pointwise on $[a,b]$" means that for each x_0 in $[a,b]$

+4 and each $\varepsilon > 0$, there corresponds an integer $N = N(x_0, \varepsilon) \ge 1$ such that $|f_n(x_0) - f(x_0)| < \varepsilon$ for all $n \ge N$.

(b) "$\{f_n\}$ converges to f uniformly on $[a,b]$" means that for each $\varepsilon > 0$ there

+4 corresponds an integer $N = N(\varepsilon) \ge 1$ (independent of x in $[a,b]$) such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N$ and all $x \in [a,b]$.

(c) Let $f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le 1/n, \\ 1 & \text{if } \frac{1}{n} < x \le 1. \end{cases}$ $(n = 1,2,3,\ldots)$



+5 Then $\{f_n\}$ converges pointwise on $[0,1]$ to the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \le 1, \end{cases}$$

but the convergence is clearly not uniform on $[0,1]$.

(d) Let $\mathcal{A}$ be an algebra of real continuous functions on a compact metric

+6 space (X, d). If $\mathcal{A}$ separates points on X and if $\mathcal{A}$ vanishes at no point

#2(g) continued.

$$\Lambda(f) = 3f(1/2) - \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx$$

$$= \int_0^1 f(x)\,d(3H(x-\tfrac{1}{2})) - \int_0^1 f(x)\,d(2\sqrt{x})$$

$$= \int_0^1 f(x)\,d\left(3H(x-\tfrac{1}{2}) - 2\sqrt{x}\right).$$

Therefore $\Lambda(f) = \int_0^1 f\,d\alpha$ for all $f \in C[0,1]$ where

+3 here $$\alpha(x) = 3H(x-\tfrac{1}{2}) - 2\sqrt{x} \qquad (0 \le x \le 1)$$

is of bounded variation on $[0,1]$.

of $\mathfrak{X}$, then to each real continuous function f on $\mathfrak{X}$ and each $\varepsilon > 0$ there corresponds a function h in $\mathcal{Q}$ such that $|f(x) - h(x)| < \varepsilon$ for all x in $\mathfrak{X}$.

+6 (e) I would show that $\mathcal{B}$ is an algebra of real continuous functions on R (a compact subset of the metric space $\mathbb{R}^2$ with the Euclidean metric) which separates points on R and vanishes at no point of R. I would then apply the Stone-Weierstrass approximation theorem.

3 pts. to here. <u>#4</u>. *Proof:* Enumerate E, say $E = \{e_j\}_{j=1}^{\infty}$. Since $\{f_n(e_1)\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, there exists a subsequence of $\{f_n\}$, which we denote by $\{f_{1,k}\}_{k=1}^{\infty}$, such that $\{f_{1,k}(e_1)\}$

6 o here. converges as $k \to \infty$. Since $\{f_{1,k}(e_2)\}_{k=1}^{\infty}$ is bounded, there exists a subsequence of $\{f_{1,k}\}$, which we denote by $\{f_{2,j}\}_{j=1}^{\infty}$, such that $\{f_{2,j}(e_2)\}$ converges as $j \to \infty$. Continuing in this manner we generate sequences

$$S_1: \quad f_{1,1} \quad f_{1,2} \quad f_{1,3} \quad \cdots$$

12 pts. to here.

$$S_2: \quad f_{2,1} \quad f_{2,2} \quad f_{2,3} \quad \cdots$$

$$S_3: \quad f_{3,1} \quad f_{3,2} \quad f_{3,3} \quad \cdots$$

$$\vdots$$

with the following properties:

(i) S_n is a subsequence of S_{n-1} for $n = 2, 3, 4, \ldots$;

(ii) for each $n \geq 1$, $\{f_{n,k}(e_n)\}$ converges as $k \to \infty$;

(iii) the order in which functions appear in S_n is the same as their relative order in S_{n-1} for $n \geq 2$.

20 pts. to here.

Consider now the "diagonal" sequence S: $f_{1,1}$ $f_{2,2}$ $f_{3,3}$ ...

By property (iii), the sequence S, except for possibly its first $n-1$ terms, is a subsequence of S_n, for $n = 1, 2, 3, \ldots$ Therefore, property (ii) implies that $\{f_{n,n}(e_i)\}_{n=1}^{\infty}$ converges as $n \to \infty$ for every $e_i \in E$.

25 pts. to here.

Math 315 Midterm Exam

Spring 2007

mean: 71.3

median: 74

standard deviation: 19.8

Distribution of Scores:

80 – 100	A	4
60 – 79	B	3
40 – 59	C	3