Real Analysis Comprehensive Examination

(Mathematics 415-416)

for Mr. Suman Sanyal

September 2007

This is a take-home examination consisting of eight problems which are to be solved within 48 hours and returned to Dr. Grow. All eight problems are of equal value and a score of 70% or higher will be required in order to receive a passing grade.

I, the undersigned, attest that all work on this examination is mine alone, that I received aid from no animate sources, and that all inanimate sources I consulted have been duly noted by me on each problem solution.

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1. Let $p_0, p_1, \ldots, p_n$ be real numbers such that each $p_i > 1$ and

$$\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p_0}.$$

If, for each integer i between 1 and n, f_i belongs to $L^{p_i}(a,b)$, must it be the case that the product $f_1 \ldots f_n$ belongs to $L^{p_0}(a,b)$? Justify your answer.

2. Let f be an absolutely continuous function on $[0,1]$ such that $f(0) = f(1)$ and $f' \in L^2[0,1]$. Show that the Fourier transforms of f' and f are related by $\hat{f'}(n) = 2\pi i n \hat{f}(n)$ for all $n \in \square$ and that $\hat{f} \in \ell^1(\square)$.

3. Let E be a Lebesgue measurable subset of $\square$. Show that

$$\lim_{\varepsilon \to 0^+} \frac{m\left(E \cap (x-\varepsilon, x+\varepsilon)\right)}{2\varepsilon}$$

is 1 a.e. on E and 0 a.e. on the complement of E.

4. Let $f \in L^p(0,\infty)$ for some $p \in [1,\infty)$ and define

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \quad \text{for } 0 < x < \infty.$$

(a) Show that to each $p \in (1,\infty)$ there corresponds a positive constant A_p such that

$$\|F\|_p \le A_p \|f\|_p \quad \text{for all } f \in L^p(0,\infty).$$

(b) What is the best constant A_p in the inequality of part (a)? Justify your answer.

(c) Does the analogue of the inequality in part (a) hold for functions in $L^1(0,\infty)$? Justify your answer.

5. In this problem, denote $\ln(0) = -\infty$ and $\exp(-\infty) = 0$. Let $f \in L^p(0,1)$ for some $p > 0$. Prove or disprove:

$$\lim_{p \to 0^+} \|f\|_p = \exp\left(\int_0^1 \ln|f(x)|\,dx\right).$$

6. Let m denote Lebesgue measure on $(0,\infty)$. For any Lebesgue measurable subset E of $\square$ define

$$\mu_1(E) = \sum_{n=1}^{\infty} \frac{1}{n^3} \int_{E \cap [n,n+1)} x\,dm,$$

$$\mu_2(E) = \int_{E \cap [1,\infty)} \frac{1}{x^2}\,dm.$$

Is m absolutely continuous with respect to μ_2? Is μ_2 absolutely continuous with respect to μ_1? Explain why or why not, and find the corresponding Radon-Nikodym derivatives, if they exist.

7. Let (X,Σ,μ) be a finite measure space, and let S denote the set of (μ-equivalence classes of) measurable real functions on X. For f and g in S, define

$$d(f,g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\, d\mu.$$

Show that d is a metric on S and that $f_n \to f$ in this metric if and only if $f_n \to f$ in measure.

8. (a) If f and g are Lebesgue measurable real functions on $\mathbb{R}$, show that the function ϕ defined by $\phi(x,y) = f(x-y)g(y)$ is a measurable function on $\mathbb{R}^2$ equipped with two-dimensional Lebesgue measure.

(b) If f and g are functions in $L^1(\mathbb{R})$, define the function $f*g$ on $\mathbb{R}$ by

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Show that $f*g$ belongs to $L^1(\mathbb{R})$, $\|f*g\|_1 \le \|f\|_1 \|g\|_1$, and the Fourier transforms are related by $(f*g)^\wedge(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$ for all real numbers ξ.

(c) Let p and q be nonnegative extended real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, show that $f*g$ is continuous on $\mathbb{R}$, that $\|f*g\|_u \le \|f\|_p \|g\|_q$, and that $(f*g)(x) \to 0$ as $|x| \to \infty$.