Sec. 1.1, pp. 4-5

#2 (a) $\mathcal{L}(u+v) = (u+v)_x + x(u+v)_y = u_x + v_x + xu_y + xv_y$

$= u_x + xu_y + v_x + xv_y = \mathcal{L}(u) + \mathcal{L}(v)$

$\mathcal{L}(ku) = (ku)_x + x(ku)_y = k(u_x + xu_y) = k\,\mathcal{L}(u)$

Linear

(b) $\mathcal{L}(u+v) = (u+v)_x + (u+v)(u+v)_y = u_x + v_x + (u+v)(u_y + v_y)$

$= u_x + uu_y + v_x + vv_y + uv_y + vu_y$

$= \mathcal{L}(u) + \mathcal{L}(v) + \underbrace{uv_y + vu_y}_{\text{not always zero}}$

Nonlinear

(c) $\mathcal{L}(u+v) = (u+v)_x + (u+v)_y^2 = u_x + v_x + (u_y + v_y)^2$

$= u_x + u_y^2 + v_x + v_y^2 + 2u_y v_y$

$= \mathcal{L}(u) + \mathcal{L}(v) + \underbrace{2u_y v_y}_{\text{not always zero}}$

Nonlinear

(d) $\mathcal{L}(u+v) = (u+v)_x + (u+v)_y + 1$

$= u_x + u_y + 1 + v_x + v_y + 1 - 1$

$= \mathcal{L}(u) + \mathcal{L}(v) \underbrace{-1}_{\text{not zero}}$

Nonlinear

(e) $\mathcal{L}(u+v) = \sqrt{1+x^2}\cos(y)(u+v)_x + (u+v)_{yxy} - \left[\arctan\left(\frac{x}{y}\right)\right](u+v)$

$= \sqrt{1+x^2}\cos(y)u_x + u_{yxy} - \left[\arctan\left(\frac{x}{y}\right)\right]u + \sqrt{1+x^2}\cos(y)v_x + v_{yxy} - \left[\arctan\left(\frac{x}{y}\right)\right]v$

$= \mathcal{L}(u) + \mathcal{L}(v)$

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#2(e) (cont.) $\mathcal{L}(ku) = \sqrt{1+x^2}\cos(y)(ku)_x + (ku)_{yxy} - \left[\arctan\left(\frac{x}{y}\right)\right](ku)$

$$= k\left\{\sqrt{1+x^2}\cos(y)u_x + u_{yxy} - \left[\arctan\left(\frac{x}{y}\right)\right]u\right]$$

$$= k\,\mathcal{L}(u)$$

$\therefore$ $\mathcal{L}$ is linear.

#3 (a) $u_t - u_{xx} + 1 = 0$ Second order, linear, inhomogeneous

(b) $u_t - u_{xx} + xu = 0$ Second order, linear, homogeneous

(c) $u_t - u_{xxt} + uu_x = 0$ Third order, nonlinear

(d) $u_{tt} - u_{xx} + x^2 = 0$ Second order, linear, inhomogeneous

(e) $iu_t - u_{xx} + \frac{1}{x}u = 0$ Second order, linear, homogeneous

(f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$ First order, nonlinear

(g) $u_x + e^y u_y = 0$ First order, linear, homogeneous

(h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$ Fourth order, nonlinear

#5 (a) $V_1 = \{[a,b,c] \in \mathbb{R}^3 : b = 0\}$ is a vector space because it is a subset of the vector space $\mathbb{R}^3$ which is closed under the operations of addition of vectors and scalar multiplication:

$$[a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2] \in V_1,$$

$$k[a, 0, c] = [ka, 0, kc] \in V_1 .$$

#5 (b) $V_2 = \{ [a,b,c] \in \mathbb{R}^3 : b = 1 \}$ is not a vector space because it is not closed under addition of vectors:

$$[a_1, 1, c_1] + [a_2, 1, c_2] = [a_1 + a_2, \overset{!}{2}, c_1 + c_2] \notin V_2 .$$

(c) $V_3 = \{ [a,b,c] \in \mathbb{R}^3 : ab = 0 \}$ is not a vector space because it is not closed under addition of vectors:

$$[0, 1, c_1] + [1, 0, c_2] = [1, 1, c_1 + c_2] \notin V_3 .$$

(d) $V_4 = \{ k_1 [1,1,0] + k_2 [2,0,1] \in \mathbb{R}^3 : k_1, k_2 \in \mathbb{R} \}$ is a vector space because it is a subset of the vector space $\mathbb{R}^3$ which is closed under the operations of addition of vectors and scalar multiplication:

$$k_1[1,1,0] + k_2[2,0,1] + k_1'[1,1,0] + k_2'[2,0,1] = \underbrace{(k_1 + k_1')[1,1,0] + (k_2 + k_2')[2,0,1]}_{\in V_4},$$

$$k\left(k_1[1,1,0] + k_2[2,0,1]\right) = kk_1[1,1,0] + kk_2[2,0,1] \in V_4 .$$

(e) $V_5 = \{ [a,b,c] \in \mathbb{R}^3 : c - a = 2b \}$ is a vector space because it is a subset of the vector space $\mathbb{R}^3$ which is closed under the operations of addition of vectors and scalar multiplication:

$$\left[a_1, \frac{c_1 - a_1}{2}, c_1\right] + \left[a_2, \frac{c_2 - a_2}{2}, c_2\right] = \left[a_1 + a_2, \frac{c_1 + c_2 - (a_1 + a_2)}{2}, c_1 + c_2\right] \in V_5 ,$$

$$k\left[a, \frac{c-a}{2}, c\right] = \left[ka, \frac{kc - ka}{2}, kc\right] \in V_5 .$$

Sec. 1.1, pp. 4-5.

#7 Suppose that $c_1(1+x) + c_2(1-x) + c_3(1+x+x^2) = 0$ for some constants c_1, c_2, c_3. Then

$$(c_1+c_2+c_3) + (c_1-c_2+c_3)x + c_3x^2 = 0,$$

and so $c_1+c_2+c_3 = c_1-c_2+c_3 = c_3 = 0$. (The only polynomial equal to zero for arbitrary x is the one whose coefficients all vanish.) It follows that $c_3=0$ and $c_1+c_2 = 0 = c_1-c_2$. From the last string of equalities, we have (upon addition of equations) that $2c_1=0$. Hence $c_1=0$ and consequently $c_2=0$ as well. We have shown that the only choice of constants c_1, c_2, and c_3 such that

$$c_1(1+x) + c_2(1-x) + c_3(1+x+x^2) = 0$$

is the trivial choice $c_1=c_2=c_3=0$. Therefore $1+x$, $1-x$, and $1+x+x^2$ are linearly independent functions.

#9 $V = \{c_1 + c_2\sin^2(x) + c_3\cos^2(x) : c_1, c_2, c_3 \in \mathbb{R}\}$ is a vector space because it is a subset of the vector space of all functions on $\mathbb{R}$ which is closed under addition of functions and scalar multiplication:

$$c_1 + c_2\sin^2(x) + c_3\cos^2(x) + c_1' + c_2'\sin^2(x) + c_3'\cos^2(x)$$

$$= (c_1+c_1') + (c_2+c_2')\sin^2(x) + (c_3+c_3')\cos^2(x) \quad \in V,$$

$$k[c_1+c_2\sin^2(x)+c_3\cos^2(x)] = kc_1 + kc_2\sin^2(x) + kc_3\,\text{os}^2(x) \quad \in V.$$

#9 (cont.) Using the Pythagorean identity $\sin^2(x)+\cos^2(x)=1$, we see that

$$c_1 + c_2\sin^2(x) + c_3\cos^2(x) = (c_1+c_3) + (c_2-c_3)\sin^2(x).$$

Because the functions 1 and $\sin^2(x)$ are linearly independent, it follows that V is two-dimensional and $\{1, \sin^2(x)\}$ is a basis for V.

#10 Let $S = \{u : u''' - 3u'' + 4u = 0\}$, and let $u_1, u_2 \in S$, $k \in \mathbb{R}$. Then $u_1''' - 3u_1'' + 4u_1 = 0$ and $u_2''' - 3u_2'' + 4u_2 = 0$. Adding equations and using linearity of differentiation, we find

$$(u_1+u_2)''' - 3(u_1+u_2)'' + 4(u_1+u_2) = 0.$$

That is, $u_1+u_2 \in S$. Likewise,

$$(ku_1)''' - 3(ku_1)'' + 4(ku_1) = k\big(\overbrace{u_1''' - 3u_1'' + 4u_1}^{0}\big) = 0$$

so $ku_1 \in S$. Thus S is a subset of the vector space of thrice differentiable functions which is closed under addition of functions and scalar multiplication, and consequently S is a vector space.

To find a basis for S, consider $u = e^{mx}$ where m is a constant to be determined so that $u = e^{mx}$ belongs to S. Then

$$m^3e^{mx} - 3m^2e^{mx} + 4e^{mx} = u''' - 3u'' + 4u = 0$$

so (dividing through by e^{mx}) we have $m^3 - 3m^2 + 4 = 0$. But

#10 (cont.) this factors as $(m-2)(m^2-m-2)=0$ or $(m-2)(m-2)(m+1)=0$. Thus $m=-1$ or $m=2$ (double root). It follows from elementary ODE theory that

$$u=(c_1+c_2x)e^{2x}+c_3e^{-x}$$

is the general solution of $u'''-3u''+4u=0$ (as one can easily check). Thus S is three-dimensional with basis $\{e^{2x}, xe^{2x}, e^{-x}\}$.

#11 Let f and g be differentiable functions of one variable and let $u(x,y)=f(x)g(y)$. Then $u_x=f'(x)g(y)$, $u_y=f(x)g'(y)$, and $u_{xy}=f'(x)g'(y)$ so

$$uu_{xy}-u_xu_y=f(x)g(y)f'(x)g'(y)-f'(x)g(y)f(x)g'(y)=0;$$

that is, $u(x,y)=f(x)g(y)$ is a solution of the PDE $uu_{xy}=u_xu_y$.

#12 Let $n>0$ and consider $u_n(x,y)=\sin(nx)\sinh(ny)$. Then $(u_n)_x=n\cos(nx)\sinh(ny)$, $(u_n)_{xx}=-n^2\sin(nx)\sinh(ny)$, $(u_n)_y=n\sin(nx)\cosh(ny)$, $(u_n)_{yy}=n^2\sin(nx)\sinh(ny)$. Consequently $(u_n)_{xx}+(u_n)_{yy}=-n^2\sin(nx)\sinh(ny)+n^2\sin(nx)\sinh(ny)=0$.