Sec. 1.2, pp. 9–10.

#1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin(x)$ when $t = 0$.

(By change-of-coordinates.) Let $\xi = 2t + 3x$ and $\eta = -3t + 2x$. Then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = 2u_\xi - 3u_\eta ,$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = 3u_\xi + 2u_\eta .$$

Substituting these expressions into the PDE yields

$$0 = 2u_t + 3u_x = 2(2u_\xi - 3u_\eta) + 3(3u_\xi + 2u_\eta)$$

$$0 = 13u_\xi .$$

But $u_\xi = 0$ implies that u is a function of η alone:

$$u = f(\eta).$$ (Here f is any differentiable function of a single variable.)

Since $\eta = -3t + 2x$,

$$u(x,t) = f(2x - 3t).$$

When $t = 0$, we must have for all real x,

$$\sin(x) = u(x,0) = f(2x - 0) = f(2x).$$

Therefore $f(u) = \sin(u/2)$ for all real u, and hence

$$\boxed{u(x,t) = \sin\left(\frac{2x-3t}{2}\right) = \sin\left(x - \frac{3}{2}t\right)} .$$

Sec. 1.2, pp. 9–10.

#3. Solve the linear equation $(1+x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

(By geometric method) Writing the PDE in the form

$$[1+x^2, 1] \cdot \nabla u(x,y) = 0,$$

we see that $u = u(x,y)$ is constant along curves in the plane whose tangent vector at a general point (x,y) is parallel to $[1+x^2, 1]$, i.e. along curves obeying

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

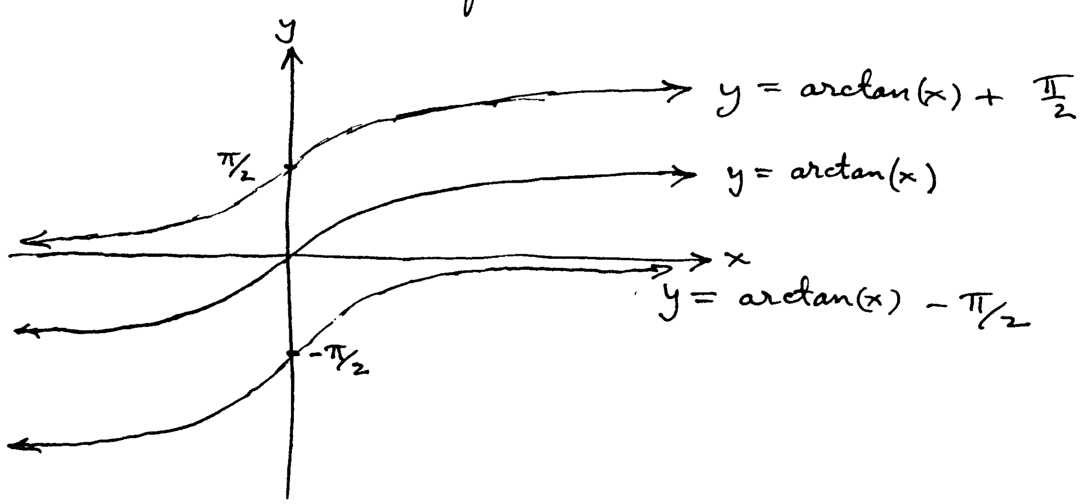
Integrating this ODE yields the characteristic curves for the PDE:

$$y = \arctan(x) + c.$$

Thus $u(x,y) = u(x, \arctan(x) + c) = u(0, \overset{0}{\overline{\arctan(0)}} + c) = f(c)$ where f is a differentiable function of a single variable. That is,

$$\boxed{u(x,y) = f(y - \arctan(x))}. \quad (\text{because } c = y - \arctan(x)).$$

Some characteristic curves for the PDE:



Sec. 1.2, pp. 9–10.

#4. Check that (7): $u(x,y) = f(e^{-x}y)$ indeed solves (4): $u_x + yu_y = 0$.

$$u_x = f'(e^{-x}y) \cdot \frac{\partial}{\partial x}(e^{-x}y) = -e^{-x}y f'(e^{-x}y)$$

$$u_y = f'(e^{-x}y) \cdot \frac{\partial}{\partial y}(e^{-x}y) = e^{-x} f'(e^{-x}y)$$

$$\therefore \quad u_x + yu_y = -e^{-x}yf'(e^{-x}y) + y\left[e^{-x}f'(e^{-x}y)\right] \overset{\checkmark}{=} 0.$$

#5. Solve the equation $\sqrt{1-x^2}\, u_x + u_y = 0$ with the condition $u(0,y) = y$.

Since the PDE has variable coefficients, we use the geometric method. Rewriting the PDE in the form $[\sqrt{1-x^2}, 1] \cdot \nabla u(x,y) = 0$ and arguing as in #3, $u = u(x,y)$ is constant al ng the characteristic curves given by

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Integrating this ODE yields $y = \arcsin(x) + c$. Thus

$$u(x,y) = u(x, \arcsin(x) + c) = u(0, \overbrace{\arcsin(0)}^{0} + c) = f(c)$$

where f is an arbitrary differentiable function of a single variable. Because $c = y - \arcsin(x)$, it follows that

$$u(x,y) = f(y - \arcsin(x)).$$

Applying the auxiliary condition, we find

$$y = u(0,y) = f(y - \overbrace{\arcsin(0)}^{0}) = f(y) \text{ for all real } y.$$

Thus $\boxed{u(x,y) = f(y - \arcsin(x)) = y - \arcsin(x).}$

Sec. 1.2, pp. 9–10.

#7. Solve $au_x + bu_y + cu = 0$.

Because of the term cu, we use the change-of-coordinates method.

Let

$$(*) \quad \begin{cases} \xi = ax + by \\ \eta = -bx + ay. \end{cases}$$

Then $u_x = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = au_\xi - bu_\eta$,

$$u_y = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = bu_\xi + au_\eta.$$

Substituting in the PDE gives

$$0 = au_x + bu_y + cu = a(au_\xi - bu_\eta) + b(bu_\xi + au_\eta) + cu$$

$$0 = (a^2+b^2)u_\xi + cu$$

$$0 = u_\xi + \left(\frac{c}{a^2+b^2}\right)u \quad \leftarrow \left(\text{Linear first-order; integrating factor is } e^{\int \frac{c}{a^2+b^2} d\xi} = e^{\frac{c\xi}{a^2+b^2}}.\right)$$

$$e^{\frac{c\xi}{a^2+b^2}} \cdot 0 = e^{\frac{c\xi}{a^2+b^2}} u_\xi + \left(\frac{c}{a^2+b^2}\right) e^{\frac{c\xi}{a^2+b^2}} u$$

$$0 = \frac{\partial}{\partial \xi}\left(e^{\frac{c\xi}{a^2+b^2}} u\right) \quad \text{(By the product rule for derivatives.)}$$

Integrating with respect to ξ, holding η fixed, gives

$$0 = e^{\frac{c\xi}{a^2+b^2}} u + k(\eta).$$

Here the "constant" of integration k may actually vary with η.

Solving for u gives

$$u = -k(\eta)e^{\frac{-c\xi}{a^2+b^2}} \overset{\text{by } (*)}{=} -k(ay-bx)e^{\frac{-c(ax+by)}{a^2+b^2}}.$$

This (correct!) solution can be rewritten to agree with the answer

#7 (cont.) in the back of the text as follows. To express $ax+by$ as a function of x and $ay-bx$, we add and subtract $\frac{b^2x}{a}$:

$$ax+by = ax + \frac{b(ay-bx)}{a} + \frac{b^2x}{a}$$

$$= \left(\frac{a^2+b^2}{a}\right)x + \frac{b}{a}(ay-bx).$$

Consequently

$$u = -k(ay-bx)e^{-\frac{c}{a^2+b^2}\left\{\overbrace{\left(\frac{a^2+b^2}{a}\right)x + \frac{b}{a}(ay-bx)}^{ax+by}\right\}}$$

$$u = -k(ay-bx)e^{-\frac{cb}{a(a^2+b^2)}(ay-bx)} \cdot e^{-\frac{cx}{a}}$$

$$\boxed{u = f(bx-ay)e^{\frac{-cx}{a}}}$$

where $f(t) = -k(-t)e^{\frac{cbt}{a(a^2+b^2)}}$.

#8. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

Because of the terms u and e^{x+2y}, we use the change-of-coordinates method. Let

$$(*) \quad \begin{cases} \xi = x + y \\ \eta = -x + y. \end{cases}$$

Then $u_x = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi - u_\eta,$

$u_y = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = u_\xi + u_\eta.$

#8 (cont.) Solving the system (*) for x and y in terms of ξ and η, we find $x = \frac{\xi-\eta}{2}$ and $y = \frac{\xi+\eta}{2}$. Substituting these expressions into the original PDE yields

$$(u_\xi - u_\eta) + (u_\xi + u_\eta) + u = e^{\frac{\xi-\eta}{2} + 2\left(\frac{\xi+\eta}{2}\right)}$$

$$u_\xi + \frac{1}{2}u = \frac{1}{2}e^{\frac{+\eta}{2}} \cdot e^{\frac{3\xi}{2}}$$ ← (Linear first-order in ξ; integrating factor $e^{\frac{\xi}{2}}$.)

$$e^{\frac{\xi}{2}}u_\xi + \frac{1}{2}e^{\frac{\xi}{2}}u = \frac{1}{2}e^{\frac{\eta}{2}} \cdot e^{2\xi}$$

$$\frac{\partial}{\partial \xi}\left(u\, e^{\frac{\xi}{2}}\right) = \frac{1}{2}e^{\frac{\eta}{2}} \cdot e^{2\xi}.$$ (Product rule for derivatives.)

Integrating with respect to ξ, holding η fixed, we have

$$u e^{\frac{\xi}{2}} = \frac{1}{4}e^{\frac{\eta}{2}}e^{2\xi} + f(\eta)$$

where, as in #7, the "constant" of integration f may actually vary with η. Solving for u yields

$$u = \frac{1}{4}e^{\frac{3\xi}{2} + \frac{\eta}{2}} + f(\eta)e^{-\frac{\xi}{2}}.$$

Substituting from (*) produces

$$u(x,y) = \frac{1}{4}e^{\frac{3}{2}(x+y) + \frac{1}{2}(y-x)} + f(y-x)e^{-\frac{1}{2}(x+y)}$$

$$= \frac{1}{4}e^{x+2y} + f(y-x)e^{-\frac{1}{2}(x+y)}.$$

Using the auxiliary condition, we have

$$0 = u(x,0) = \frac{1}{4}e^{x+0} + f(0-x)e^{-\frac{1}{2}(x+0)} = \frac{e^x}{4} + f(-x)e^{-\frac{x}{2}},$$

for all real x.

#8 (cont.) Solving for f, we have $-\frac{1}{4}e^{\frac{3x}{2}} = f(-x)$ for all real x,

i.e. $f(t) = -\frac{1}{4}e^{-\frac{3t}{2}}$ for $-\infty < t < \infty$. Consequently

$$u(x,y) = \frac{1}{4}e^{x+2y} + f(y-x)e^{-\frac{1}{2}(x+y)}$$

$$= \frac{1}{4}e^{x+2y} + \left(-\frac{1}{4}\right)e^{\frac{-3(y-x)}{2}} \cdot e^{-\frac{1}{2}(x+y)}$$

$$= \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}$$

$$\boxed{u(x,y) = \frac{1}{4}e^{x}\left(e^{2y} - e^{-2y}\right) = \frac{1}{2}e^{x}\sinh(2y)}$$

#9. Solve $au_x + bu_y = f(x,y)$ where $f = f(x,y)$ is a given (continuous) function. Write the solution in the form

$$(\dagger) \qquad u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\,ds + g(bx-ay)$$

where g is an arbitrary (differentiable) function of one variable, L is the characteristic line segment from the y-axis to the point (x,y), and the integral is a line integral.

(We use the method of change-of-coordinates.) Let

$$(*) \quad \begin{cases} \xi = ax + by, \\ \eta = bx - ay. \end{cases}$$

Then the ξ-η is an orthogonal coordinate system with origin coinciding with that of the x-y system. Solving $(*)$ for x and y in terms of ξ and η we find

$$(**) \quad \begin{cases} x = \dfrac{\begin{vmatrix} \xi & b \\ \eta & -a \end{vmatrix}}{\begin{vmatrix} a & b \\ b & -a \end{vmatrix}} = \dfrac{-a\xi - b\eta}{-a^2-b^2} = \dfrac{a\xi + b\eta}{a^2+b^2}, \\[2ex] y = \dfrac{\begin{vmatrix} a & \xi \\ b & \eta \end{vmatrix}}{-a^2-b^2} = \dfrac{a\eta - b\xi}{-a^2-b^2} = \dfrac{b\xi - a\eta}{a^2+b^2}. \end{cases}$$

Also, the chain rule for derivatives and $(*)$ imply $u_x = au_\xi + bu_\eta$ and $u_y = bu_\xi - au_\eta$. Substituting the above expressions in the original PDE yields

Sec. 1.2, pp. 9–10.

#9 (cont.) $a(au_\xi + \cancel{bu_\eta}) + b(bu_\xi - \cancel{au_\eta}) = f\left(\frac{a\xi+b\eta}{a^2+b^2}, \frac{b\xi-a\eta}{a^2+b^2}\right)$

$$u_\xi = \frac{1}{a^2+b^2} f\left(\frac{a\xi+b\eta}{a^2+b^2}, \frac{b\xi-a\eta}{a^2+b^2}\right)$$

Holding η fixed and integrating with respect to ξ we find

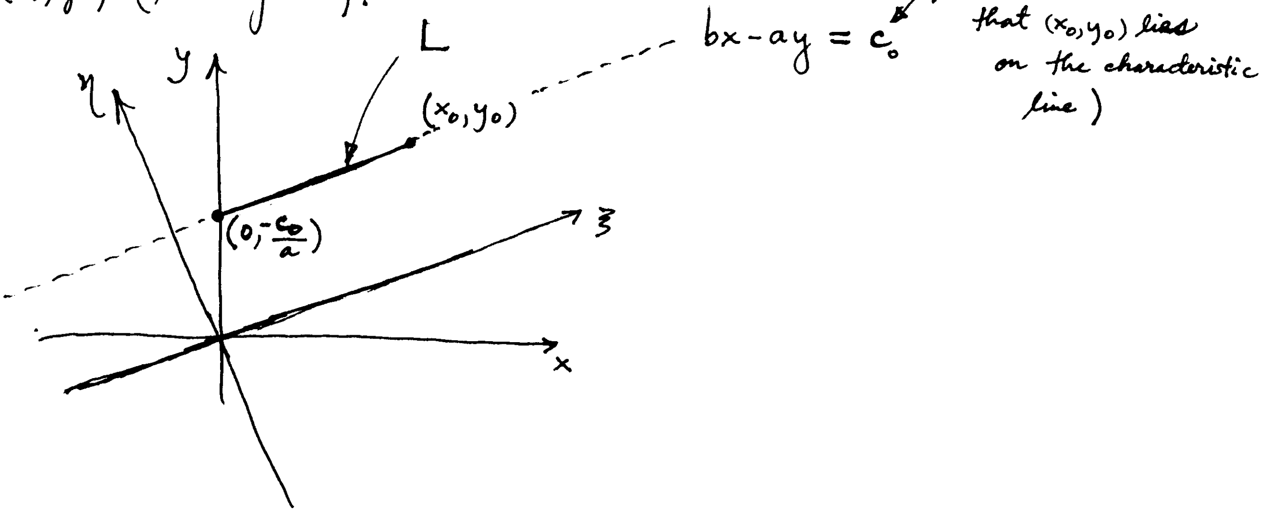
$$(***) \quad \boxed{u = u(\xi,\eta) = \int \frac{1}{a^2+b^2} f\left(\frac{a\xi+b\eta}{a^2+b^2}, \frac{b\xi-a\eta}{a^2+b^2}\right) d\xi \; + \; g(\eta)}$$

where g is an arbitrary (differentiable) function of a single variable.
We now seek to express the solution ~~(***)~~ in the form (†).

The characteristic lines of the PDE are given by

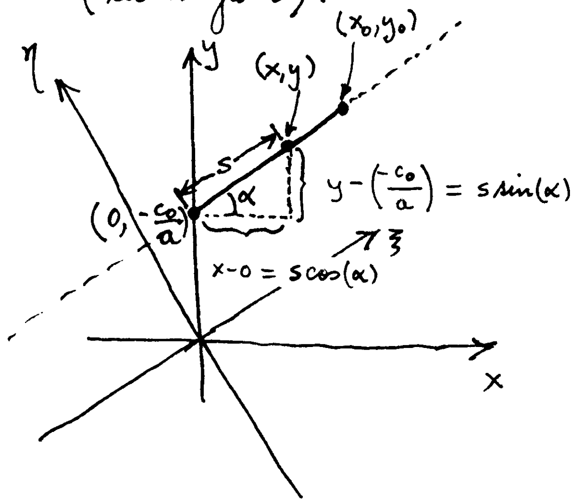
$$(****) \qquad \frac{dy}{dx} = \frac{b}{a}.$$

Integrating we find that the characteristic lines are $(\eta =)\; bx - ay = c$ (see (*))
where c is an arbitrary constant. Given an arbitrary point (x_0, y_0) in the plane, let L denote the characteristic line segment from the y-axis to (x_0, y_0) (see diagram).



An equation (in parametric form) for L in terms of the arclength s of the

#9 (cont.) segment along L from $(0, -\frac{c_0}{a})$ to (x, y) is obtained as follows (see diagram).



$$\tan(\alpha) = \frac{dy}{dx} = \frac{b}{a} \text{ by } (****),$$

so $\sec(\alpha) = \sqrt{1 + \tan^2(\alpha)} = \sqrt{1 + \left(\frac{b}{a}\right)^2}$

and hence $\cos(\alpha) = \dfrac{1}{\sqrt{1 + \left(\frac{b}{a}\right)^2}} = \dfrac{a}{\sqrt{a^2+b^2}}$.

Also $\sin(\alpha) = \tan(\alpha)\cos(\alpha) = \dfrac{b}{a} \cdot \dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{b}{\sqrt{a^2+b^2}}$.

Therefore

$$\boxed{\begin{aligned} x &= s\cos(\alpha) = \frac{as}{\sqrt{a^2+b^2}} \\ y &= \frac{-c_0}{a} + s\sin(\alpha) = -\frac{c_0}{a} + \frac{bs}{\sqrt{a^2+b^2}} \end{aligned}}$$

and

Note that $s = 0$ corresponds to the point $(x,y) = (0, -\frac{c_0}{a})$, the left endpoint of L, and that $s = x_0\sqrt{1+\left(\frac{b}{a}\right)^2}$ corresponds to the right endpoint of L:

$$(x,y) = \left(\frac{a x_0 \sqrt{1+(b/a)^2}}{\sqrt{a^2+b^2}},\ -\frac{c_0}{a} + \frac{b x_0 \sqrt{1+(b/a)^2}}{\sqrt{a^2+b^2}} \right)$$

$$= \left(x_0,\ -\frac{c_0}{a} + \frac{b x_0}{a} \right)$$

$$= (x_0, y_0) \qquad (\text{since } bx_0 - ay_0 = c_0).$$

For future reference, note that using $(**)$, we have at the right endpoint of L:

$$s = x_0\sqrt{1+(b/a)^2} = \left(\frac{a\xi_0 + b\eta_0}{a^2+b^2}\right)\sqrt{1+\left(\frac{b}{a}\right)^2} = \left(\frac{\xi_0 + \frac{b\eta_0}{a}}{a^2+b^2}\right)\sqrt{a^2+b^2} = \frac{\xi_0 + \frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}.$$

#9 (cont.) Therefore
$$\frac{1}{\sqrt{a^2+b^2}}\int_L f\,ds = \frac{1}{\sqrt{a^2+b^2}}\int_0^{\frac{\xi_0+\frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}} f\Big(\overbrace{\frac{as}{\sqrt{a^2+b^2}}}^{x(s)},\ \overbrace{-\frac{c_0}{a}+\frac{bs}{\sqrt{a^2+b^2}}}^{y(s)}\Big)ds$$

satisfies

$$\frac{\partial}{\partial\xi_0}\left\{\frac{1}{\sqrt{a^2+b^2}}\int_0^{\frac{\xi_0+\frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}} f\Big(\frac{as}{\sqrt{a^2+b^2}},\ -\frac{c_0}{a}+\frac{bs}{\sqrt{a^2+b^2}}\Big)ds\right\}$$

(2nd Fundamental Theorem of Calculus plus chain rule)

$$= \frac{1}{\sqrt{a^2+b^2}} f\left(\frac{a\left(\frac{\xi_0+\frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}\right)}{\sqrt{a^2+b^2}},\ -\frac{c_0}{a}+\frac{b\left(\frac{\xi_0+\frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}\right)}{\sqrt{a^2+b^2}}\right)\cdot\frac{\partial}{\partial\xi_0}\left(\frac{\xi_0+\frac{b\eta_0}{a}}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{1}{a^2+b^2} f\left(\frac{a\xi_0+b\eta_0}{a^2+b^2},\ -\frac{c_0}{a}+\frac{b\xi_0+\frac{b^2}{a}\eta_0}{a^2+b^2}\right)$$

$$= \frac{1}{a^2+b^2} f\left(\frac{a\xi_0+b\eta_0}{a^2+b^2},\ \frac{-\frac{c_0}{a}(a^2+\cancel{b^2})+b\xi_0+\cancel{\frac{b^2}{a}\eta_0}}{a^2+b^2}\right) \qquad (\text{since } \eta_0=c_0)$$

$$= \frac{1}{a^2+b^2} f\left(\frac{a\xi_0+b\eta_0}{a^2+b^2},\ \frac{b\xi_0-a\eta_0}{a^2+b^2}\right).$$

Hence $\frac{1}{\sqrt{a^2+b^2}}\int_L f\,ds$ is, as a function of ξ, an antiderivative of $\frac{1}{a^2+b^2} f\left(\frac{a\xi+b\eta}{a^2+b^2}, \frac{b\xi-a\eta}{a^2+b^2}\right)$, and consequently may be substituted for the integral on the right hand side of (***):

$$u = \frac{1}{\sqrt{a^2+b^2}}\int_L f\,ds + g(\eta).$$

Making use of (*), we see that the solution above can be written in the form (†).

Sec. 1.2, pp. 9–10.

#11. Use the coordinate method to solve the equation $u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2$.

Let $\begin{cases} \xi = x + 2y, \\ \eta = 2x - y. \end{cases}$

Then $u_x = u_\xi + 2u_\eta$ and $u_y = 2u_\xi - u_\eta$, so substituting in the PDE yields

$$(u_\xi + \cancel{2u_\eta}) + 2(2u_\xi - \cancel{u_\eta}) + \eta u = (2x-y)(x+2y) = \eta\xi$$

$$u_\xi + \tfrac{1}{5}\eta u = \tfrac{1}{5}\eta\xi \longleftarrow \left(\begin{array}{l}\text{Linear first-order, integrating} \\ \text{factor } e^{\int \frac{1}{5}\eta\, d\xi} = e^{\frac{1}{5}\eta\xi}.\end{array}\right)$$

$$e^{\eta\xi/5} u_\xi + \tfrac{1}{5}\eta e^{\eta\xi/5} u = \tfrac{1}{5}\eta\xi e^{\eta\xi/5}$$

$$\frac{\partial}{\partial\xi}\left(e^{\eta\xi/5} u \right) = \tfrac{1}{5}\eta\xi e^{\eta\xi/5}$$

$$e^{\eta\xi/5} u = \int \tfrac{1}{5}\eta\xi e^{\eta\xi/5}\, d\xi$$

But integration by parts shows that $\int axe^{ax}\,dx = (x - \frac{1}{a})e^{ax} + c$. Therefore

$$e^{\eta\xi/5} u = (\xi - \tfrac{5}{\eta}) e^{\eta\xi/5} + f(\eta)$$

$$u = \xi - \frac{5}{\eta} + e^{-\eta\xi/5} f(\eta)$$

$$u = x + 2y - \frac{5}{2x-y} + e^{-\frac{1}{5}(2x^2+3xy-2y^2)} f(2x-y)$$

where f is an arbitrary (differentiable) function of a single variable.