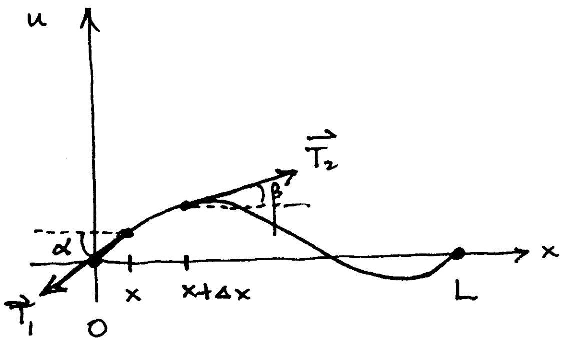
Sec. 1.3, pp. 18-19.

#1. Carefully derive the equation of a string in a medium in which resistance is proportional to the velocity.

Let $u = u(x,t)$ denote the vertical displacement of the string at horizontal position x and time t. Fix the time $t>0$ and consider the segment of string between x and $x+\Delta x$. (See diagram below.)



Applying Newton's second law of motion to this segment we have:

(horizontal component) $$0 = \overbrace{|\vec{T}_2|\cos\beta - |\vec{T}_1|\cos\alpha}^{\text{net horizontal tension}}$$

(vertical component) $$\int_x^{x+\Delta x} \underbrace{\rho(\xi)d\xi}_{\text{mass}}\, \underbrace{u_{tt}(\xi,t)}_{\substack{\text{acceleration}\\ \text{(vertical)}}} = \underbrace{|\vec{T}_2|\sin\beta - |\vec{T}_1|\sin\alpha}_{\text{net vertical tension}} - \underbrace{\int_x^{x+\Delta x} k u_t(\xi,t)d\xi}_{\text{resistance}}$$

As in the derivation of the one-dimensional wave equation without resistance,

$$\cos\beta = \frac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}} \qquad \cos\alpha = \frac{1}{\sqrt{1+u_x^2(x,t)}}$$

$$\sin\beta = \frac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}} \qquad \sin\alpha = \frac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}.$$

If we neglect the u_x^2-term under the radical in the denominators, then

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#1 (cont.)
$$0 = |\vec{T}_1| \cdot 1 - |\vec{T}_2| \cdot 1 \quad \text{so} \quad |\vec{T}| = \text{constant} = T_0$$

and
$$\int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi = T_0\left[u_x(x+\Delta x,t) - u_x(x,t)\right] - \int_x^{x+\Delta x} k u_t(\xi,t)\,d\xi .$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, we have

$$\rho(x) u_{tt}(x,t) = T_0\, u_{xx}(x,t) - k u_t(x,t)$$

provided ρ, u_t, and u_{tt} are continuous at x. If the linear density is constant, i.e. $\rho(x) = \rho_0$ for all $0 \le x \le L$ then

$$u_{tt}(x,t) - \frac{T_0}{\rho_0} u_{xx}(x,t) + \frac{k}{\rho_0} u_t(x,t) = 0,$$

i.e.
$$\boxed{u_{tt} - c^2 u_{xx} + r u_t = 0}$$

where $c = \sqrt{\dfrac{T_0}{\rho_0}}$ and $r = \dfrac{k}{\rho_0}$ are positive constants.

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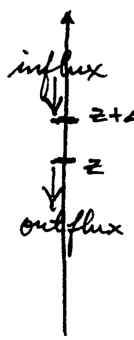
\#4. Suppose that some particles which are suspended in a liquid medium would be pulled down at the constant velocity $V > 0$ by gravity in the absence of diffusion. Taking account of diffusion, find the equation for the concentration of particles. Assume homogeneity in the horizontal directions x and y. Let the z-axis point upwards.

Let $u = u(z,t)$ denote the concentration of particles (per unit length) at vertical position z and time t. The total amount of particles between z and $z+\Delta z$ at a (fixed) time $t > 0$ is

$$M = \int_z^{z+\Delta z} u(\xi,t)\,d\xi .$$



By Theorem 1, Appendix A.3, p. 390, the time rate of change of the amount particles between z and $z+\Delta z$ at time t is

$$\frac{dM}{dt} = \frac{d}{dt}\int_z^{z+\Delta z} u(\xi,t)\,d\xi = \int_z^{z+\Delta z} u_t(\xi,t)\,d\xi .$$

The loss or gain of particles occurs only at the boundaries z and $z+\Delta z$; thus

$$\frac{dM}{dt} = \underset{\text{at } z+\Delta z}{\text{influx of particles}} - \underset{\text{at } z}{\text{outflux of particles}}$$

$$= \Big[\overbrace{k u_z(z+\Delta z,t)}^{\text{Fick's law}} + V u(z+\Delta z,t)\Big] - \Big[\overbrace{k u_z(z,t)}^{\text{Fick's law}} + V u(z,t)\Big].$$

Equating the two expressions for dM/dt above and dividing by Δz yields

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} u_t(\xi,t)\,d\xi = k\left[\frac{u_z(z+\Delta z,t) - u_z(z,t)}{\Delta z}\right] + V\left[\frac{u(z+\Delta z,t) - u(z,t)}{\Delta z}\right].$$

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#4. (cont.) Letting $\Delta z \to 0$, we have

$$u_t(z,t) = k\,u_{zz}(z,t) + V u_z(z,t),$$

i.e. $$\boxed{u_t - k u_{zz} - V u_z = 0}$$

#7. Consider heat flow in a ball where the temperature depends only on the time t and the (radial) spherical coordinate $r = \sqrt{x^2+y^2+z^2}$ from the center $(x,y,z) = (0,0,0)$ of the ball. Derive the equation

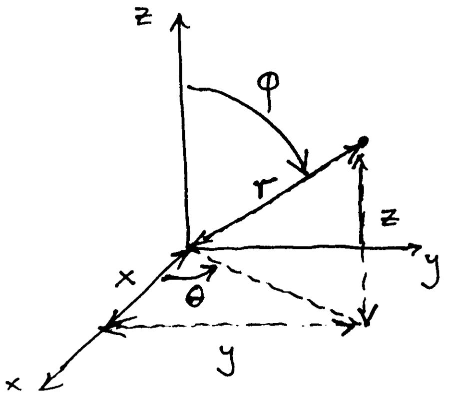
$$(*) \qquad u_t = k\left(u_{rr} + \frac{2}{r}u_r\right).$$

Let $u = u(x,y,z,t)$ be the temperature at position (x,y,z) in the ball and at time t. From (10) p.16 we know that

$$(**) \qquad u_t = k\nabla^2 u = k(u_{xx} + u_{yy} + u_{zz}) \qquad (k \text{ a constant} > 0)$$

when the specific heat, mass density, and heat conductivity are constant. However, in spherical coordinates (see diagram below) the Laplacian operator ∇^2 acting on $u = u(r,\varphi,\theta)$ is equal to

$$(***) \qquad \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}.$$



$z = r\cos\varphi$

$y = r\sin\varphi\sin\theta$

$x = r\sin\varphi\cos\theta$

#7. (cont.) In the problem at hand the function u depends only on r and is independent of φ and θ; thus $\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial \theta} = 0$ at all points in the ball. Consequently, from (***) we obtain

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

or

$$\nabla^2 u = \frac{1}{r^2}\left(r^2 u_{rr} + 2r u_r\right) = u_{rr} + \frac{2}{r} u_r .$$

Substituting this expression for $\nabla^2 u$ into (**) produces (*).

#9. This is an exercise on the divergence theorem

$$(\dagger) \qquad \iiint_D \nabla \cdot \vec{F} \, d\vec{x} = \iint_{\partial D} \vec{F} \cdot \vec{n} \, dS$$

valid for any bounded domain D in space with boundary surface ∂D and unit outward normal vector $\vec{n}$. If you never learned it, see Section A.3. As an exercise, verify it in the following case by calculating both sides separately: $\vec{F} = r^2 \vec{x}$, $\vec{x} = x\vec{i} + y\vec{j} + z\vec{k}$, $r^2 = x^2 + y^2 + z^2$, and D = the ball of radius a and center at the origin.

We compute the left member of $(\dagger)$ first:

$$\vec{F} = r^2 \vec{x} = (x^2+y^2+z^2)(x\vec{i} + y\vec{j} + z\vec{k}) = (x^2+y^2+z^2)x\vec{i} + (x^2+y^2+z^2)y\vec{j} + (x^2+y^2+z^2)z\vec{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)x\right] + \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)y\right] + \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)z\right]$$

$$= \left[(x^2+y^2+z^2) + 2x^2\right] + \left[(x^2+y^2+z^2) + 2y^2\right] + \left[(x^2+y^2+z^2) + 2z^2\right]$$

$$= 5(x^2+y^2+z^2)$$

$$= 5r^2$$

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#9. (cont.) In spherical coordinates (see diagram in problem #7),

$$D = \{(r;\varphi;\theta): \ 0 \le r < a, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\}$$

$$d\vec{x} = r^2 \sin\varphi \, dr \, d\varphi \, d\theta .$$

Therefore

$$\iiint_D \nabla \cdot \vec{F} \, d\vec{x} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 5r^2 \cdot r^2 \sin\varphi \, dr \, d\varphi \, d\theta = \left(r \Big|_0^a\right)\left(-\cos\varphi \Big|_0^{\pi}\right)\left(\theta \Big|_0^{2\pi}\right)$$

$$= 4\pi a^5 .$$

We now compute the right member of (†); for points on

$$\partial D = \{(r;\varphi;\theta): \ r = a, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\},$$

we have $\vec{n} = \dfrac{\vec{x}}{r} = \dfrac{1}{a}\vec{x}$,

and $dS = a^2 \sin\varphi \, d\varphi \, d\theta$. Therefore

$$\iint_{\partial D} \vec{F} \cdot \vec{n} \, dS = \int_0^{2\pi}\int_0^{\pi} a^2 \vec{x} \cdot \frac{1}{a}\vec{x} \, a^2 \sin\varphi \, d\varphi \, d\theta$$

$$= a^3 \int_0^{2\pi}\int_0^{\pi} \overbrace{\vec{x}\cdot\vec{x}}^{a^2} \sin\varphi \, d\varphi \, d\theta$$

$$= a^5 \left(-\cos\varphi \Big|_0^{\pi}\right)\left(\theta \Big|_0^{2\pi}\right)$$

$$= 4\pi a^5 .$$

This verifies (†) in the case at hand.

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\# 10. If $\vec{f}(\vec{x})$ is continuous and $|\vec{f}(\vec{x})| \le \dfrac{1}{|\vec{x}|^3+1}$ for all $\vec{x}$, show that

$$(*) \qquad \iiint_{\text{all space}} \nabla\cdot\vec{f}\, d\vec{x} = 0 .$$

Let $D = D(a;\vec{0})$ be the ball of radius a and center at the origin $\vec{0}$. Then

$$\left| \iiint_D \nabla\cdot\vec{f}\, d\vec{x} \right| \overset{\text{Divergence Theorem}}{=} \left| \iint_{\partial D} \vec{f}\cdot\vec{n}\, dS \right| \le \iint_{\partial D} |\vec{f}\cdot\vec{n}|\, dS$$

$$\le \iint_{\partial D} |\vec{f}|\,|\vec{n}|\, dS \le \iint_{\partial D} \frac{1}{a^3+1}\cdot 1\, dS$$

$$= \left(\frac{1}{a^3+1}\right)\overbrace{\left(4\pi a^2\right)}^{\text{surface area of sphere of radius } a}$$

Therefore $\displaystyle \left| \iiint_{\text{all space}} \nabla\cdot\vec{f}\, d\vec{x} \right| = \lim_{a\to\infty} \left| \iint_{D(a;\vec{0})} \vec{f}\cdot\vec{n}\, dS \right| \le \lim_{a\to\infty} \frac{4\pi a^2}{a^3+1} = 0,$

and hence $(*)$ follows.