Sec. 4.3, pp. 97 – 100.

#1. Find the eigenvalues graphically for the problem

$$\begin{cases} X'' + \lambda X = 0 & \text{for } 0 < x < l \\ X(0) = 0, \quad X'(l) + aX(l) = 0 . \end{cases}$$

Assume that $a \neq 0$.

Case $\lambda = 0$: The general solution to the ODE is $X(x) = c_1 x + c_2$. The B.C.'s yield:

$$0 = X(0) = c_1 \cdot 0 + c_2 \Rightarrow c_2 = 0$$

$$0 = X'(l) + aX(l) = c_1 + a(c_1 l + \cancelto{0}{c_2}) = c_1(1 + al) .$$

Therefore $\boxed{\lambda = 0 \text{ is an eigenvalue if and only if } a = -\frac{1}{l}.}$ In this case $X(x) = x$ is a corresponding eigenfunction.

Case $\lambda > 0$, say $\lambda = -\beta^2$: The general solution to the ODE is $X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. The B.C.'s yield:

$$0 = X(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$$

$$0 = X'(l) + aX(l) = \left[-\beta \cancelto{0}{c_1} \sin(\beta l) + \beta c_2 \cos(\beta l)\right] + a\left[\cancelto{0}{c_1} \cos(\beta l) + c_2 \sin(\beta l)\right]$$

$$\Rightarrow 0 = c_2\left[\beta \cos(\beta l) + a \sin(\beta l)\right]$$

$$c_2 \neq 0 \Rightarrow \beta \cos(\beta l) = -a \sin(\beta l)$$

Subcase $\cos(\beta l) = 0$: Then $\sin(\beta l) = \pm 1$ and hence $a = 0$, a contradiction.

Subcase $\cos(\beta l) \neq 0$: Then $-\frac{\beta}{a} = \tan(\beta l)$

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#1. (cont.) $\quad \ell = \left.\frac{d}{d\beta}(\tan\beta\ell)\right|_{\beta=0} < \left.\frac{d}{d\beta}\left(-\frac{1}{a}\beta\right)\right|_{\beta=0} = -\frac{1}{a}.$

Since this inequality is possible only when $a<0$, we get $\boxed{0>a>-\frac{1}{\ell}}$ as the condition for an eigenvalue $\lambda_0 \in \left(0, \left(\frac{\pi}{2\ell}\right)^2\right)$. Notice that an eigenfunction corresponding to $\lambda_n = \beta_n^2$ is

$$X_n(x) = \sin(\beta_n x).$$

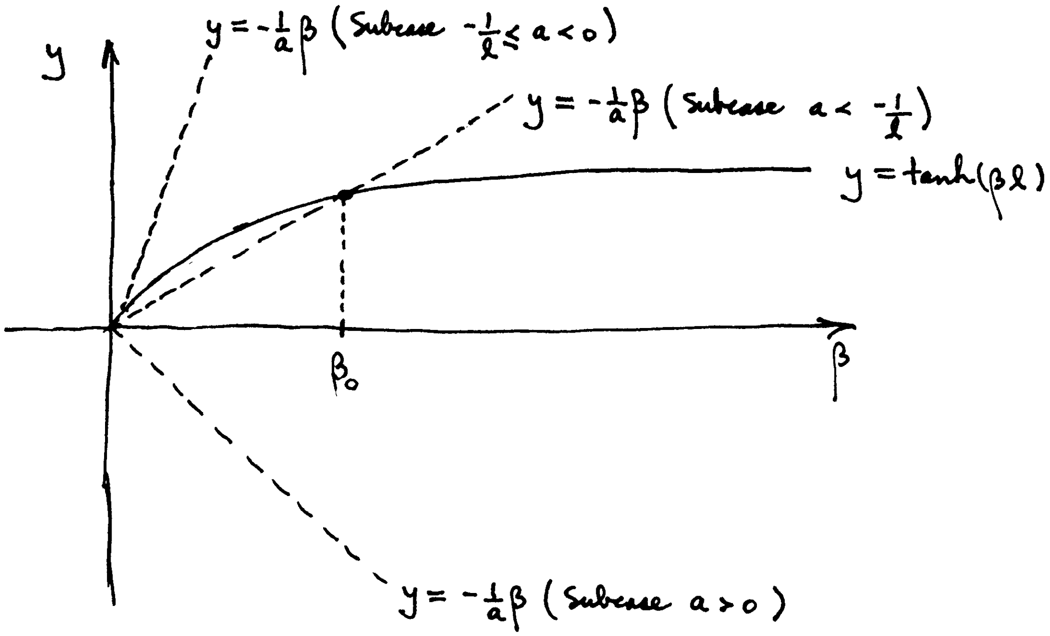
Case $\lambda<0$, say $\lambda = -\beta^2$: The general solution to the ODE is $X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x)$. The B.C.'s yield

$$0 = X(0) = c_1\cdot 1 + c_2\cdot 0 = c_1,$$

$$0 = X'(\ell) + aX(\ell) = \beta \overset{0}{\cancel{c_1}}\sinh(\beta\ell) + \beta c_2\cosh(\beta\ell) + a\left[\overset{0}{\cancel{c_1}}\cosh(\beta\ell) + c_2\sinh(\beta\ell)\right]$$

$$= c_2\left[\beta\cosh(\beta\ell) + a\sinh(\beta\ell)\right],$$

and $c_2 \neq 0$ implies $-\frac{\beta}{a} = \tanh(\beta\ell)$.



#1. (cont.) We clearly have either $\boxed{\text{one or no negative eigenvalues.}}$ If the slope of $y = \tanh(\beta l)$ at 0 is greater than that of $y = -\frac{1}{a}\beta$ at $\underbrace{0}_{(\text{and } a<0)}$, then there will be one negative eigenvalue, i.e.

$$l = \frac{d}{d\beta}\left(\tanh(\beta l)\right)\Big|_{\beta=0} > \frac{d}{d\beta}\left(-\frac{1}{a}\beta\right)\Big|_{\beta=0} = -\frac{1}{a}.$$

Thus $al < -1$ (since $a<0$), which is equivalent to $\boxed{a < -\frac{1}{l}, \text{ is the condition for one negative eigenvalue.}}$ $\boxed{\text{If either } a>0 \text{ or } -\frac{1}{l} \le a < 0 \text{ then there are no negative eigenvalues.}}$ Notice that $X_0(x) = \sinh(\beta_0 x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_0 = -(\beta_0)^2$.

See the next page for a summary of the results for the eigenvalue problem in #1, p. 97.

Summary for #1, p. 97, Sec. 4.3

For the eigenvalue problem $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \quad X'(l) + aX(l) = 0 \end{cases}$ $(a \neq 0)$

then we have:

	Eigenvalues	Eigenfunctions
If $a > 0$	$\lambda_n = \beta_n^2 \in \left(\frac{(n-\frac{1}{2})^2\pi^2}{l^2}, \frac{n^2\pi^2}{l^2} \right)$ $(n = 1, 2, 3, \ldots)$ such that $\lambda_n - \frac{(n-\frac{1}{2})^2\pi^2}{l^2} \to 0$ as $n \to \infty$	$X_n(x) = \sin(\beta_n x)$
If $0 > a > -\frac{1}{l}$	$\lambda_n = \beta_n^2 \in \left(\frac{n^2\pi^2}{l^2}, \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \right)$ $(n = 1, 2, 3, \ldots)$ such that $\lambda_n - \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \to 0$ as $n \to \infty$. Plus, $\lambda_0 = \beta_0^2 \in \left(0, \frac{\pi^2}{4l^2}\right)$.	$X_n(x) = \sin(\beta_n x)$
If $a = -\frac{1}{l}$	$\lambda_0 = 0$; $\lambda_n = \beta_n^2 \in \left(\frac{n^2\pi^2}{l^2}, \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \right)$ $(n = 1, 2, 3, \ldots)$ such that $\lambda_n - \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \to 0$ as $n \to \infty$	$X_0(x) = x$; $X_n(x) = \sin(\beta_n x)$
If $a < -\frac{1}{l}$	$\lambda_0 = -\beta_0^2 < 0$; $\lambda_n = \beta_n^2 \in \left(\frac{n^2\pi^2}{l^2}, \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \right)$ $(n = 1, 2, 3, \ldots)$ such that $\lambda_n - \frac{(n+\frac{1}{2})^2\pi^2}{l^2} \to 0$ as $n \to \infty$	$X_0(x) = \sinh(\beta_0 x)$; $X_n(x) = \sin(\beta_n x)$

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#2. Consider the eigenvalue problem with Robin B.C.'s at both ends:

$$(\square)\quad \begin{cases} -X''(x) = \lambda X(x) & \text{for } 0<x<\ell, \\ X'(0) - a_0 X(0) = 0, \\ X'(\ell) + a_\ell X(\ell) = 0. \end{cases}$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_\ell = -a_0 a_\ell \ell$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue.

(a) Suppose that $\lambda=0$ is an eigenvalue. Then there exist constants c_0 and c_1 (not both zero) such that $X(x) = c_1 x + c_0$ is a solution to the ODE $-X''(x) = 0$ on $(0,\ell)$ and satisfies the Robin B.C.'s above. The B.C.'s imply

$$c_1 - a_0 c_0 = X'(0) - a_0 X(0) = 0,$$

$$c_1 + a_\ell(c_1 \ell + c_0) = X'(\ell) + a_\ell X(\ell) = 0.$$

But this means that $\theta = c_1$, $\tau = c_0$ is a nontrivial solution to the linear homogeneous system of equations

$$(\dagger)\quad \begin{cases} \theta - a_0\tau = 0, \\ (1+\ell a_\ell)\theta + a_\ell \tau = 0. \end{cases}$$

Consequently (by Cramer's Rule, say), the determinant of the system $(\dagger)$ must vanish:

$$1\cdot a_\ell - (1+\ell a_\ell)\cdot(-a_0) = 0.$$

Rearranging yields the desired relation:

$$(*)\qquad a_0 + a_\ell = -a_0 a_\ell \ell.$$

Conversely, suppose that the relation $(*)$ holds. Then the

#2. (cont.) determinant of the coefficient matrix of (†) vanishes and consequently there exist (infinitely many, in fact!) nontrivial solutions $\theta = c_1$, $\tau = c_0$ to (†). But then $X(x) = c_1 x + c_0$ is a nontrivial solution to $-X''(x) = 0$ on $(0, \ell)$ satisfying $X'(0) - a_0 X(0) = X'(\ell) + a_\ell X(\ell) = 0$. That is, $\lambda = 0$ is an eigenvalue of (□).

(b) From the work in part (a), it is apparent that $X(x) = c_1 x + c_0$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$ provided $\theta = c_1$, $\tau = c_0$ is a nontrivial solution to (†). Thus $X(x) = a_0 x + 1$ is an eigenfunction.

#3. Derive the eigenvalue equation

$$(16) \qquad \tanh(\gamma \ell) = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}$$

for the negative eigenvalues $\lambda = -\gamma^2$ to the problem

$$(*) \quad \begin{cases} -X'' = \lambda X, \\ X'(0) - a_0 X(0) = 0, \\ X'(\ell) + a_\ell X(\ell) = 0. \end{cases}$$

Also, derive the formula

$$(17) \qquad X(x) = \cosh(\gamma x) + \frac{a_0}{\gamma} \sinh(\gamma x)$$

for the corresponding eigenfunctions.

#3. (cont.) The general solution to the ODE $X'' - \gamma^2 X = 0$ is $X(x) = c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x)$. The B.C.'s yield:

$$\begin{cases} 0 = X'(0) - a_0 X(0) = \gamma c_2 - a_0 c_1, \\ 0 = X'(\ell) + a_\ell X(\ell) = c_1 \gamma \sinh(\gamma \ell) + c_2 \gamma \cosh(\gamma \ell) + a_\ell [c_1 \cosh(\gamma \ell) + c_2 \sinh(\gamma \ell)]. \end{cases}$$

Rearranging, we have the system

$$(\square) \begin{cases} -a_0 c_1 + \gamma c_2 = 0, \\ [\gamma \sinh(\gamma \ell) + a_\ell \cosh(\gamma \ell)] c_1 + [\gamma \cosh(\gamma \ell) + a_\ell \sinh(\gamma \ell)] c_2 = 0. \end{cases}$$

The existence of a nontrivial solution to this linear homogeneous system (which is equivalent to saying that $\lambda = -\gamma^2$ is an eigenvalue of $(*)$) implies that the determinant of the coefficient matrix vanishes:

$$-a_0[\gamma \cosh(\gamma \ell) + a_\ell \sinh(\gamma \ell)] - \gamma[\gamma \sinh(\gamma \ell) + a_\ell \cosh(\gamma \ell)] = 0.$$

Simplifying produces

$$\gamma (a_0 + a_\ell) \cosh(\gamma \ell) = -[a_0 a_\ell + \gamma^2] \sinh(\gamma \ell).$$

Because $\gamma \neq 0$, it follows that $\gamma \cosh(\gamma \ell) \neq 0$ and $\sinh(\ell \gamma) \neq 0$. Therefore either $a_0 + a_\ell = 0 = a_0 a_\ell + \gamma^2$ or (16) holds. That is, either $\gamma^2 = -a_0 a_\ell = a_0^2$ or

$$(16) \qquad \tanh(\gamma \ell) = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}.$$

Note that in either case, the first equation of $(\square)$ implies $c_2 = \frac{a_0 c_1}{\gamma}$, so the corresponding eigenfunction is

$$X(x) = c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x) = c_1 \left(\cosh(\gamma x) + \frac{a_0}{\gamma} \sinh(\gamma x) \right).$$

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#4. Consider the Robin eigenvalue problem:

$$(*)\begin{cases} -X'' = \lambda X & \text{for } 0<x<\ell, \\ X'(0) - a_0 X(0) = X'(\ell) + a_\ell X(\ell) = 0 & . \end{cases}$$

If $a_0 < 0$, $a_\ell < 0$, and $-a_0 - a_\ell < a_0 a_\ell \ell$, show that there are two negative eigenvalues. This case may be called "substantial absorption at both ends".

Suppose that λ is a negative eigenvalue of $(*)$, say $\lambda = -\gamma^2$. By problem #3, γ must satisfy the equation

$$(16) \qquad \tanh(\gamma \ell) = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}.$$

The function $y = \tanh(\gamma \ell)$ satisfies the following relations:

(a) $y(0) = 0$.

(b) $y'(\gamma) = \ell \operatorname{sech}^2(\gamma \ell) > 0$ so $y \uparrow$ on $(0,\infty)$ and $y'(0) = \ell$.

(c) $\lim_{\gamma \to \infty} y(\gamma) = 1$.

The function $y = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}$, on the other hand, has the following properties:

(a′) $y(0) = 0$.

(b′) $y'(\gamma) = -\frac{(a_0 + a_\ell)(a_0 a_\ell - \gamma^2)}{(\gamma^2 + a_0 a_\ell)^2}$ so $y \uparrow$ on $(0, \sqrt{a_0 a_\ell})$,

$y \downarrow$ on $(\sqrt{a_0 a_\ell}, +\infty)$, and $y'(0) = -\frac{(a_0 + a_\ell)}{a_0 a_\ell}$.

(c′) $\lim_{\gamma \to \infty} y(\gamma) = 0$.

(d′) $y(\sqrt{a_0 a_\ell}) = -\frac{(a_0 + a_\ell)}{2\sqrt{a_0 a_\ell}} \geq 1$.

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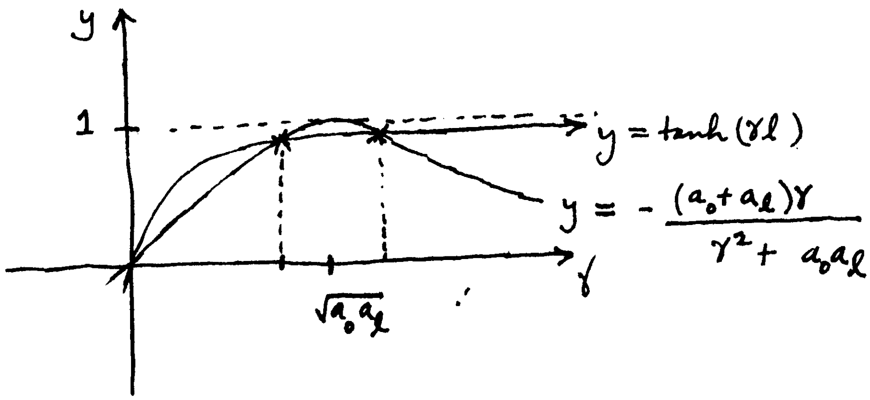
#4 (cont.) To see that property (d′) holds, we appeal to the following lemma: Let x and y be positive real numbers. Then $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof:
$$\begin{aligned}(x+y)^2 - (2\sqrt{xy})^2 &= x^2+2xy+y^2-4xy \\ &= x^2-2xy+y^2 \\ &= (x-y)^2 \\ &\geq 0.\end{aligned}$$

Therefore $(x+y)^2 \geq (2\sqrt{xy})^2$ and hence $x+y \geq 2\sqrt{xy}$.



From the hypotheses $a_0<0$, $a_\ell<0$, and $-(a_0+a_\ell) < a_0a_\ell \ell$, and properties (b) and (b′), it follows that the slope of $y=\tanh(\gamma\ell)$ at $\gamma=0$ exceeds the slope of $y = \frac{-(a_0+a_\ell)\gamma}{\gamma^2+a_0a_\ell}$ at $\gamma=0$:

$$\frac{d}{d\gamma}\left[\tanh(\gamma\ell)\right]\Big|_{\gamma=0} = \ell > -\frac{(a_0+a_\ell)}{a_0a_\ell} = \frac{d}{d\gamma}\left[-\frac{(a_0+a_\ell)\gamma}{\gamma^2+a_0a_\ell}\right]\Bigg|_{\gamma=0}.$$

Therefore using (a) and (a′), it follows that $\frac{-(a_0+a_\ell)\gamma}{\gamma^2+a_0a_\ell} < \tanh(\gamma\ell)$ for all $\gamma \in (0,\varepsilon)$ for some $\varepsilon>0$. By (b), (c), and (d′) therefore, there

#4 (cont.) exists a solution to (16) in the interval $0 < \gamma < \sqrt{a_0 a_\ell}$. Furthermore (c′), (d′), (b), and (c) imply that there exists another solution to (16) in the interval $\sqrt{a_0 a_\ell} < \gamma < +\infty$.

#5. In exercise 4 (substantial absorption at both ends) show graphically that there is an infinite number of positive eigenvalues. Show graphically that they satisfy

$$(11) \qquad \left(\frac{n\pi}{\ell}\right)^2 < \lambda_n < \left[\frac{(n+1)\pi}{\ell}\right]^2, \qquad (n = 1, 2, 3, \dots)$$

and

$$(12) \qquad \lim_{n\to\infty} \left[\lambda_n - \left(\frac{n\pi}{\ell}\right)^2\right] = 0 .$$

Following the same line of reasoning as on p. 91, we find that positive eigenvalues $\lambda = \beta^2 > 0$ of the Robin eigenvalue problem in exercise 4 must satisfy the equation obtained by dividing (7), p. 91, by C:

$$(7') \qquad (a_0 + a_\ell)\cos(\beta\ell) = \left(\beta - \frac{a_0 a_\ell}{\beta}\right)\sin(\beta\ell).$$

From (7′) it follows that either $\beta = \sqrt{a_0 a_\ell}$ or

$$(10') \qquad \frac{\beta(a_0 + a_\ell)}{\beta^2 - a_0 a_\ell} = \tan(\beta\ell).$$

The analysis given in case 1, p. 92 and figure 1, p. 92 is very nearly the same as for (10′) since $a_0 a_\ell > 0$. The

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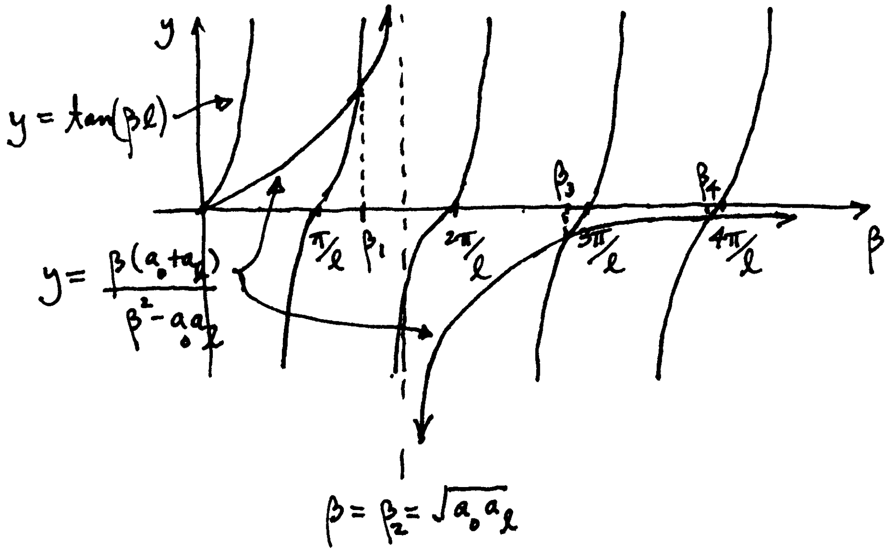
#5 (cont.) only difference is that $a_0 + a_\ell < 0$.



Note that

$$\frac{d}{d\beta}\left[\tan\beta\ell\right]\Big|_{\beta=0} = \ell > -\frac{(a_0+a_\ell)}{a_0 a_\ell} = \frac{d}{d\beta}\left[\frac{\beta(a_0+a_\ell)}{\beta^2 - a_0 a_\ell}\right]\Bigg|_{\beta=0}$$

so the slope of $y = \tan(\beta\ell)$ at $\beta = 0$ exceeds the slope of $y = \dfrac{\beta(a_0+a_\ell)}{\beta^2 - a_0 a_\ell}$ at $\beta = 0$, and hence there is no intersection of the two curves in the interval $(0, \frac{\pi}{\ell})$.

Graphically it is apparent that there exists an infinite sequence of intersection points $\beta_n \in \left(\frac{n\pi}{\ell}, \frac{(n+1)\pi}{\ell}\right)$, $(n = 1, 2, 3, \ldots)$, satisfying $\lim_{n\to\infty}\left[\beta_n - \frac{n\pi}{\ell}\right] = 0$. Since $\lambda_n = \beta_n^2$, (11) and (12) follow.

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#6. If $a_0 = a_\ell = a$ in the Robin eigenvalue problem, i.e.

$$(*)\begin{cases} -X''(x) = \lambda X(x) & \text{for } 0<x<\ell, \\ X'(0) - aX(0) = X'(\ell) + aX(\ell) = 0, \end{cases}$$

show that:

(a) There are no negative eigenvalues if $a \geq 0$, there is one if $-\frac{2}{\ell} \leq a < 0$, and there are two if $a < -\frac{2}{\ell}$.

(b) Zero is an eigenvalue if $a=0$ or $a = -\frac{2}{\ell}$.

(b) By exercise #2, $\lambda = 0$ is an eigenvalue of the Robin problem if and only if $a_0 + a_\ell = -a_0 a_\ell \ell$. When $a_0 = a_\ell = 0$, this condition becomes $2a = -a^2\ell$, or equivalently $(2+a\ell)a = 0$. Thus $\lambda = 0$ is an eigenvalue of $(*)$ if and only if either $a=0$ or $a = -\frac{2}{\ell}$.

(a) If $a_\ell = a_0 = a < -\frac{2}{\ell}$ then $-(a_0+a_\ell) = -2a < a^2\ell = a_0 a_\ell \ell$. Therefore exercise #4 implies that $(*)$ has two negative eigenvalues.

If $a_\ell = a_0 = a \geq 0$ then the analysis for the negative eigenvalue equation

$$(16) \qquad \tanh(\gamma\ell) = -\frac{(a_0+a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}$$

given in case 1 (see especially figure 4, p. 95) shows that there are no negative eigenvalues for $(*)$ when $a \geq 0$.

If $a_\ell = a_0 = a$ then the equation (16), p. 94, for negative eigenvalues $\lambda = -\gamma^2$ of the Robin problem becomes

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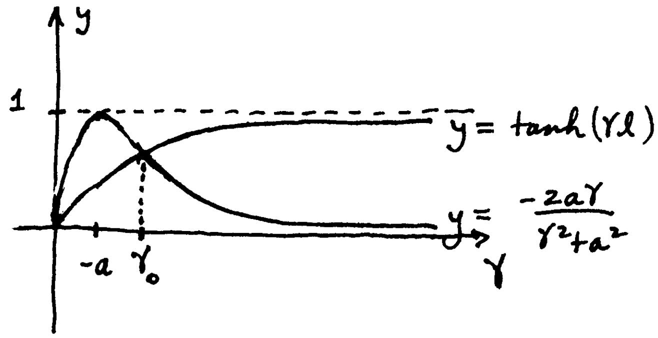
6 (a) (cont.)

$$(16') \qquad \tanh(\gamma l) = \frac{-2a\gamma}{\gamma^2+a^2}.$$

If $-\frac{2}{l} \le a < 0$ then the slope of $y = \tanh(\gamma l)$ at $\gamma = 0$ does not exceed the slope of $y = \frac{-2a\gamma}{\gamma^2+a^2}$ at $\gamma = 0$:

$$\frac{d}{d\gamma}\left[\tanh(\gamma l)\right]\Big|_{\gamma=0} = l \le -\frac{2}{a} = \frac{d}{d\gamma}\left[\frac{-2a\gamma}{\gamma^2+a^2}\right]\Big|_{\gamma=0}.$$



Also note that

$$\max_{0<\gamma<\infty} \frac{-2a\gamma}{\gamma^2+a^2} = \frac{-2a(-a)}{(-a)^2+a^2} = 1.$$

Consequently, there is precisely one intersection point for these two curves on the interval $0 < \gamma < +\infty$, and it occurs at $\gamma_0 > -a > 0$. Thus there is one negative eigenvalue $\lambda = -\gamma_0^2$ of (*) when $-\frac{2}{l} \le a < 0$.

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#7. If $a_0 = a_\ell = a$, show that as $a \to +\infty$, the eigenvalues of the Robin problem

$$(*) \begin{cases} -X''(x) = \lambda X(x), & 0<x<\ell, \\ X'(0) - aX(0) = 0 = X'(\ell) + aX(\ell) = 0, \end{cases}$$

tend to the eigenvalues of the Dirichlet problem

$$(**) \begin{cases} -X''(x) = \lambda X(x), & 0<x<\ell, \\ X(0) = X(\ell) = 0. \end{cases}$$

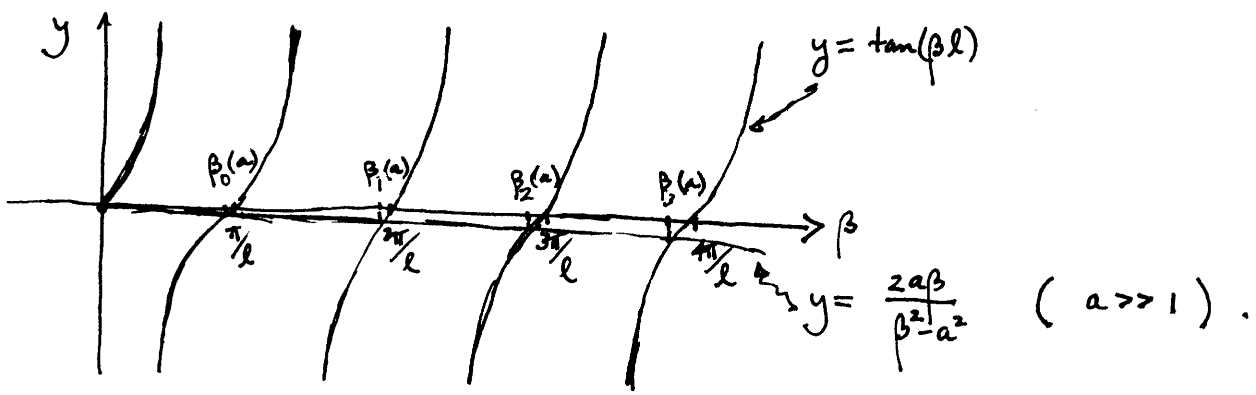
That is, (for each $n = 0, 1, 2, \ldots$),

$$(***) \lim_{a\to\infty} \left\{ \lambda_n(a) - \left[\frac{(n+1)\pi}{\ell}\right]^2 \right\} = 0.$$

The relation satisfied by the positive eigenvalues $\lambda = \beta^2$ of $(*)$ is

$$(10') \qquad \tan(\beta\ell) = \frac{2a\beta}{\beta^2 - a^2}.$$

(Set $a_0 = a_\ell = a$ in (10), p. 92.)



Clearly, $\frac{2a\beta}{\beta^2-a^2} = \frac{2(\beta/a)}{(\beta/a)^2-1} \to 0$ as $a \to \infty$. Geometrically, it is apparent that for each fixed $n = 1, 2, 3, \ldots$, the nth intersection point of the

#7 (cont.). curves $y = \tan(\beta l)$ and $y = \frac{2a\beta}{\beta^2 - a^2}$ approaches the nth β-intercept of the function $y = \tan(\beta l)$. Thus

$$(****) \quad \lim_{a\to\infty} \left\{ \beta_{n-1}(a) - \frac{n\pi}{l} \right\} = 0$$

for each $n = 1, 2, 3, \ldots$. Since $\lambda_n(a) = \beta_n^2(a)$, (***) follows from (****).

#11. (a) Prove that the (total) energy is conserved for (solutions to) the wave equation with (homogeneous) Dirichlet boundary conditions, where the energy is defined to be

$$(*) \qquad E = \frac{1}{2}\int_0^l \left[c^{-2}u_t^2(x,t) + u_x^2(x,t)\right]dx.$$

(b) Do the same for the (homogeneous) Neumann boundary conditions.

(c) For the (homogeneous) Robin boundary conditions, show that

$$E_R = \frac{1}{2}\int_0^l \left[c^{-2}u_t^2(x,t) + u_x^2(x,t)\right]dx + \frac{1}{2}a_l\left[u(l,t)\right]^2 + \frac{1}{2}a_0\left[u(0,t)\right]^2$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive, and "gained" from the boundary if a_0 and a_l are negative.

(a) Let $u = u(x,t)$ solve $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < l$, $0 < t < \infty$, subject to

$$\begin{cases} u(x,0) = \varphi(x) & \text{for } 0 \le x \le l, \\ u_t(x,0) = \psi(x) & \text{" \quad "}, \\ u(0,t) = u(l,t) & \text{for } t \ge 0. \end{cases}$$

To show that $E = E(t)$ given by (*) is constant, it suffices to show

#11(a) (cont.) that $dE/dt = 0$ for $t \geq 0$. To this end, we compute:

$$\frac{dE}{dt} = \frac{d}{dt}\left\{\frac{1}{2}\int_0^\ell \left[c^{-2}u_t^2 + u_x^2\right]dx\right\}$$

$$= \frac{1}{2}\int_0^\ell \frac{\partial}{\partial t}\left[c^{-2}u_t^2 + u_x^2\right]dx$$

$$= \int_0^\ell \left[c^{-2}u_t u_{tt} + u_x u_{xt}\right]dx .$$

But $c^{-2}u_{tt} = u_{xx}$ since $u = u(x,t)$ solves the wave equation, so

$$\frac{dE}{dt} = \int_0^\ell \left[u_t u_{xx} + u_x u_{xt}\right]dx$$

$$= \int_0^\ell (u_x u_t)_x \, dx$$

$$(\dagger) \qquad = u_x(\ell,t)u_t(\ell,t) - u_x(0,t)u_t(0,t)$$

for $t \geq 0$. Observe that since u satisfies homogeneous Dirichlet B.C.'s

$$u_t(\ell,t) = \lim_{h\to 0}\frac{u(\ell,t+h) - u(\ell,t)}{h} = \lim_{h\to 0}\left(\frac{0-0}{h}\right) = 0,$$

and similarly $u_t(0,t) = 0$ for $t \geq 0$. Therefore

$$\frac{dE}{dt} = u_x(\ell,t)\cdot 0 - u_x(0,t)\cdot 0 = 0 \qquad \text{for } t \geq 0.$$

(b) Let $u = u(x,t)$ solve $u_{tt} - c^2u_{xx} = 0$ for $0 < x < \ell$, $0 < t < \infty$ subject to

$$\begin{cases} u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } 0 \leq x \leq \ell, \\ u_x(0,t) = u_x(\ell,t) = 0 & \text{for } t \geq 0. \end{cases}$$

#11 (b) (cont.) Arguing as in part (a), we find that by (†) and the Neumann B.C.'s

$$\frac{dE}{dt} = u_x(\ell,t)u_t(\ell,t) - u_x(0,t)u_t(0,t) = 0\cdot u_t(\ell,t) - 0\cdot u_t(0,t) = 0$$

for $t \geq 0$. Hence $E = E(t)$ given by (*) is constant for $t \geq 0$.

(c) Let $u = u(x,t)$ solve $u_{tt} - c^2 u_{xx} = 0$ for $0<x<\ell$, $0<t<\infty$, subject to

$$\begin{cases} u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) \text{ for } 0 \leq x \leq \ell, \\ u_x(0,t) - a_0 u(0,t) = u_x(\ell,t) + a_\ell u(\ell,t) = 0 \quad \text{for } t \geq 0. \end{cases}$$

Arguing as in part (a), we find that by (†) and the Robin B.C.'s

$$\begin{aligned} \frac{dE}{dt} &= u_x(\ell,t)u_t(\ell,t) - u_x(0,t)u_t(0,t) \\ &= -a_\ell\, u(\ell,t)u_t(\ell,t) - a_0\, u(0,t)u_t(0,t) \\ &= -\frac{d}{dt}\left\{\frac{a_\ell}{2}[u(\ell,t)]^2 + \frac{a_0}{2}[u(0,t)]^2\right\} \end{aligned}$$

for $t \geq 0$. But $E_R = E + \frac{1}{2}a_\ell[u(\ell,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$ so the above calculation implies $\frac{dE_R}{dt} = 0$ for $t \geq 0$, i.e. $E_R = E_R(t)$ is a constant function of time $t \geq 0$.

#13. Consider a string which is fixed at the end $x=0$ and is free at the end $x=\ell$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem $u_{tt} - c^2 u_{xx} = 0$ for $0<x<\ell$, $0<t<+\infty$, $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$ for $0 \leq x \leq 1$, and $u(0,t) = u_{tt}(\ell,t) + k u_x(\ell,t) = 0$ for $t \geq 0$ and some constant k.

#13. (cont.) (b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

(b) Let $u(x,t) = X(x)T(t)$ be a nontrivial solution to the problem in part (a). Then

$$0 = u_{tt} - c^2 u_{xx} = T''X - c^2 T X''$$

so

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \text{constant} = -\lambda .$$

Therefore
$$\begin{cases} T'' + \lambda c^2 T = 0, \\ \boxed{X'' + \lambda X = 0}. \end{cases}$$

Applying the B.C.'s produces

$$0 = u(0,t) = X(0)T(t) \quad \text{for all } t \geq 0 \text{ and hence } \boxed{X(0) = 0}.$$

$$\begin{aligned} 0 = u_{tt}(l,t) + k u_x(l,t) &= X(l)T''(t) + kX'(l)T(t) \\ &= X(l)\left[-\lambda c^2 T(t)\right] + kX'(l)T(t) \\ &= \left[-\lambda c^2 X(l) + kX'(l)\right]T(t) \qquad \text{for all } t \geq 0. \end{aligned}$$

Therefore $\boxed{X'(l) - \dfrac{\lambda c^2}{k} X(l) = 0}$.

The eigenvalue problem is the boxed differential equation and the two boxed boundary conditions, one at $x=0$ and the other at $x=l$.

(c) Suppose $\lambda = \beta^2 > 0$ is a (positive) eigenvalue of the B.V.P. in part (b). The general solution of the O.D.E. $X'' + \beta^2 X = 0$ is $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$. Applying the B.C.'s yield

$$0 = X(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$$

$$0 = X'(l) - \frac{\lambda c^2}{k} X(l) = -\beta \overset{0}{\cancel{c_1}} \sin(\beta l) + \beta c_2 \cos(\beta l) - \frac{\beta^2 c^2}{k}\left[\overset{0}{\cancel{c_1}}\cos(\beta l) + c_2 \sin(\beta l)\right]$$

#13. (cont.) Since $c_2 \neq 0$, the second B.C. implies

$$\beta \cos(\beta \ell) = \frac{\beta^2 c^2}{k} \sin(\beta \ell)$$

If $\cos(\beta \ell) = 0$ then $\sin(\beta \ell) = \pm 1$ and the right member of the above equation does not vanish. This contradiction shows that $\cos(\beta \ell) \neq 0$ and consequently

$$\boxed{\frac{k}{c^2 \beta} = \tan(\beta \ell)}$$

is the equation satisfied by positive eigenvalues $\lambda = \beta^2$. Since $c_1 = 0$ and $c_2 \neq 0$, $\boxed{X(x) = \sin(\beta x)}$ is a corresponding eigenfunction.

Sec. 4.3, pp. 97–100.

#18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE

$$(*) \qquad u_{tt} + c^2 u_{xxxx} = 0 \quad \text{for } 0<x<l,\ 0<t<\infty .$$

It has initial conditions as for the wave equation. Let's say that on the end $x=0$ it is clamped (fixed), meaning that it satisfies

$$(**) \qquad u(0,t) = u_x(0,t) = 0 \quad \text{for } t \ge 0 .$$

On the other end $x=l$ it is free, meaning that it satisfies

$$(***) \qquad u_{xx}(l,t) = u_{xxx}(l,t) = 0 \quad \text{for } t\ge 0 .$$

Thus there is a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

(a) Let $u = X(x)T(t)$ be a solution to $(*)$. Substituting in $(*)$ gives

$$0 = u_{tt} + c^2 u_{xxxx} = T'' X + c^2 T X^{(iv)}$$

so

$$-\frac{T''}{c^2 T} = + \frac{X^{(iv)}}{X} = \text{constant} = \lambda .$$

Thus

$$\begin{cases} T'' + c^2 \lambda T = 0, \\ \boxed{X^{(iv)} - \lambda X = 0}. \end{cases}$$

#18. (a) (cont.) Applying the B.C.'s (**) and (***) produce:

$$0 = u(0,t) = X(0)T(t) \text{ for } t \geq 0 \text{ so } \boxed{X(0) = 0};$$

$$0 = u_x(0,t) = X'(0)T(t) \text{ for } t \geq 0 \text{ so } \boxed{X'(0) = 0};$$

$$0 = u_{xx}(l,t) = X''(l)T(t) \text{ for } t \geq 0 \text{ so } \boxed{X''(l) = 0};$$

$$0 = u_{xxx}(l,t) = X'''(l)T(t) \text{ for } t \geq 0 \text{ so } \boxed{X'''(l) = 0}.$$

The eigenvalue problem consists of the boxed O.D.E. and the four boxed B.C.'s.

(b) If $\lambda = 0$ then the general solution to $X^{(iv)} = 0$ is $X(x) = c_3x^3 + c_2x^2 + c_1x + c_0$. The B.C.'s imply

$$0 = X(0) = c_0,$$

$$0 = X'(0) = c_1,$$

$$0 = X''(l) = 6c_3 l + 2c_2,$$

$$0 = X'''(l) = 6c_3.$$

Clearly then $X(x) \equiv 0$, and this implies that zero is not an eigenvalue of the problem in part (a).

(c) Let λ be a positive eigenvalue for the problem in part (a), say $\lambda = \beta^4$ where $\beta > 0$. The general solution to $X^{(iv)} - \beta^4 X = 0$ is found as follows. Consider a trial solution of the form $X(x) = e^{mx}$. Then $m^4 e^{mx} - \beta^4 e^{mx} = X^{(iv)}(x) - \beta^4 X(x) = 0$ so $m^4 - \beta^4 = 0$. That is, $0 = (m^2 - \beta^2)(m^2 + \beta^2) = (m-\beta)(m+\beta)(m^2+\beta^2)$ and hence $\beta = m$, $-\beta = m$, $i\beta = m$, or $-i\beta = m$. The general solution is thus

$$X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x} + c_3 e^{i\beta x} + c_4 e^{-i\beta x}$$

$$= c_1' \cosh(\beta x) + c_2' \sinh(\beta x) + c_3' \cos(\beta x) + c_4' \sin(\beta x)$$

#18(c) (cont.) where the c_j's and c_j''s are arbitrary constants $(j = 1, 2, 3, 4)$.
Applying the B.C.'s we find

$$
(\dagger)\begin{cases}
0 = X(0) = c_1' + c_3' \\[4pt]
0 = X'(0) = \beta c_2' + \beta c_4' = \beta\left(c_2' + c_4'\right) \\[4pt]
0 = X''(\ell) = \beta^2 c_1' \cosh(\beta\ell) + \beta^2 c_2' \sinh(\beta\ell) - \beta^2 c_3' \cos(\beta\ell) - \beta^2 c_4' \sin(\beta\ell) \\
\qquad\quad = \beta^2\left[c_1' \cosh(\beta\ell) + c_2' \sinh(\beta\ell) - c_3' \cos(\beta\ell) - c_4' \sin(\beta\ell)\right] \\[4pt]
0 = X'''(\ell) = \beta^3 c_1' \sinh(\beta\ell) + \beta^3 c_2' \cosh(\beta\ell) + \beta^3 c_3' \sin(\beta\ell) - \beta^3 c_4' \cos(\beta\ell) \\
\qquad\quad = \beta^3\left[c_1' \sinh(\beta\ell) + c_2' \cosh(\beta\ell) + c_3' \sin(\beta\ell) - c_4' \cos(\beta\ell)\right]
\end{cases}
$$

Dividing by β, β^2, and β^3 respectively in the second, third, and fourth equations of $(\dagger)$, and then substituting the relations $c_1' = -c_3'$ and $c_2' = -c_4'$, obtained from the first two equations of $(\dagger)$, into the second pair of equations in $(\dagger)$ yields

$$
(\dagger\dagger)\begin{cases}
-\left[\cosh(\beta\ell) + \cos(\beta\ell)\right] c_3' - \left[\sinh(\beta\ell) + \sin(\beta\ell)\right] c_4' = 0 \\[6pt]
\left[\sin(\beta\ell) - \sinh(\beta\ell)\right] c_3' - \left[\cosh(\beta\ell) + \cos(\beta\ell)\right] c_4' = 0 .
\end{cases}
$$

If $\lambda = \beta^4$ is an eigenvalue then there exists a nontrivial solution to the homogeneous system of two linear equations $(\dagger)$ in the two "variables" c_3' and c_4'.
Hence the coefficient matrix of $(\dagger\dagger)$ must have a vanishing determinant:

$$
0 = \begin{vmatrix}
-\left[\cosh(\beta\ell) + \cos(\beta\ell)\right] & -\left[\sinh(\beta\ell) + \sin(\beta\ell)\right] \\[6pt]
\left[\sin(\beta\ell) - \sinh(\beta\ell)\right] & -\left[\cosh(\beta\ell) + \cos(\beta\ell)\right]
\end{vmatrix}
$$

$$
= \left[\cosh(\beta\ell) + \cos(\beta\ell)\right]^2 + \left[\sin(\beta\ell) - \sinh(\beta\ell)\right]\left[\sinh(\beta\ell) + \sin(\beta\ell)\right]
$$

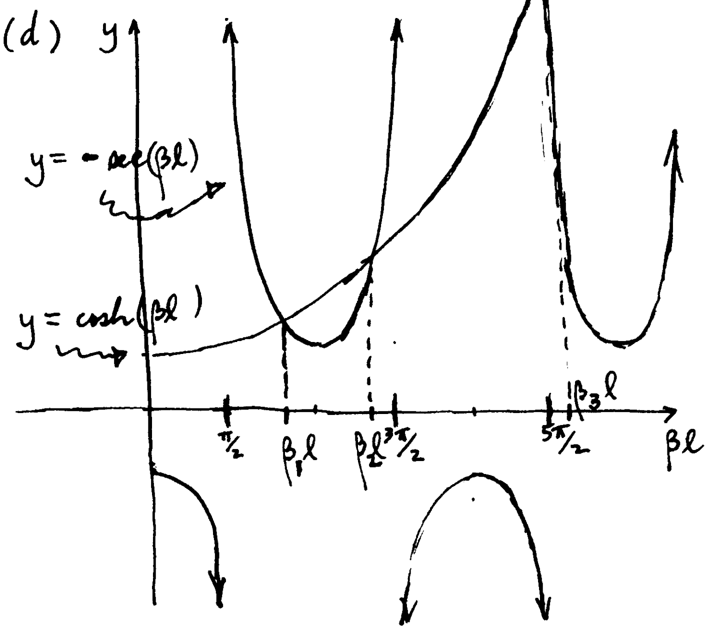
$$
= \cosh^2(\beta\ell) + 2\cosh(\beta\ell)\cos(\beta\ell) + \underbrace{\cos^2(\beta\ell) + \sin^2(\beta\ell)}_{1} - \sinh^2(\beta\ell)
$$

Sec. 4.3, pp. 97-100.

#18(c) (cont.) $\quad 0 = 2 + 2\cosh(\beta l)\cos(\beta l)$

$$\boxed{-1 = \cosh(\beta l)\cos(\beta l)}$$

(d)



$\beta_1 l \approx 1.8751$

$\beta_2 l \approx 4.694$

$\beta_3 l \approx 7.855$

By numerical methods (e.g. Newton's method).

Note that there is an infinite sequence $\{\beta_n l\}_{n=1}^{\infty}$ of intersection points, and asymptotically, we have

$$\lim_{n\to\infty}\left\{\beta_n - \frac{(2n-1)\pi}{2l}\right\} = 0.$$

(d) Let $u_n(x,t) = X_n(x)T_n(t)$ be the fundamental vibration corresponding to the nth eigenvalue $\lambda_n = \beta_n^4$ $(n=1,2,3,\ldots)$. The frequency of this vibration

#18(d) (cont.) is determined by the time-dependent factor T_n which is a solution to

$$T_n'' + c^2\lambda_n T = 0,$$

i.e. $$T_n'' + c^2\beta_n^4 T = 0.$$

Elementary techniques yield $T_n(t) = \alpha_n \cos(c\beta_n^2 t) + \beta_n \sin(c\beta_n^2 t)$ where α_n, β_n are arbitrary constants. Hence $\boxed{c\beta_n^2}$ $(n=1,2,3,\dots)$ comprise the frequencies of vibration.

(e) For the vibrating string (see (12) p.85 and the accompanying discussion) the fifth overtone has a frequency β_6 that is exactly 6 times the frequency β_1 of the fundamental tone:

$$\frac{\beta_6}{\beta_1} = \frac{\dfrac{6\pi\sqrt{T}}{l\sqrt{\rho}}}{\dfrac{\pi\sqrt{T}}{l\sqrt{\rho}}} = 6.$$

The first overtone of the tuning fork has a frequency $c\beta_2^2$ which is higher (6.26667 times as great versus 6 times as great) relative to the fundamental tone than the fifth overtone of the string.

$$\frac{c\beta_2^2}{c\beta_1^2} = \frac{\beta_2^2 l^2}{\beta_1^2 l^2} \approx \frac{(4.694)^2}{(1.8751)^2} \approx 6.26667$$