Section 5.3, p. 118

#2. (a) On the interval $[-1,1]$, show that the function x is orthogonal to the constant functions.

(b) Find a quadratic polynomial that is orthogonal to both 1 and x (on $[-1,1]$).

(c) Find a cubic polynomial that is orthogonal to all quadratics (on $[-1,1]$).

(These are the first few Legendre polynomials.)

(a) It suffices to show that x and 1 are orthogonal on $[-1,1]$:

$$\langle x, 1\rangle = \int_{-1}^{1} x \cdot 1\, dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$$

(b) A general quadratic polynomial has the form $p(x) = ax^2 + bx + c$ where $a, b,$ and c are constants with $a \neq 0$. Notice that

$$\langle p, 1\rangle = a\langle x^2 + \tfrac{b}{a}x + \tfrac{c}{a}, 1\rangle$$

$$\langle p, x\rangle = a\langle x^2 + \tfrac{b}{a}x + \tfrac{c}{a}, x\rangle$$

so it is enough to find a monic polynomial $q(x) = x^2 + \beta x + \gamma$ which is orthogonal to 1 and x on $[-1,1]$. We want $\langle q, 1\rangle = \langle q, x\rangle = 0$ on $[-1,1]$. This leads to the following two equations that the coefficients β and γ of q must satisfy:

$$0 = \langle q, 1\rangle = \int_{-1}^{1} (x^2 + \beta x + \gamma)\cdot 1\, dx = \left.\left(\frac{x^3}{3} + \frac{\beta x^2}{2} + \gamma x\right)\right|_{-1}^{1} = \frac{2}{3} + 2\gamma$$

$$0 = \langle q, x\rangle = \int_{-1}^{1} (x^2 + \beta x + \gamma)x\, dx = \left.\left(\frac{x^4}{4} + \frac{\beta x^3}{3} + \frac{\gamma x^2}{2}\right)\right|_{-1}^{1} = \frac{2\beta}{3}$$

Therefore $\beta = 0$ and $\gamma = -1/3$, i.e. $\boxed{q(x) = x^2 - \frac{1}{3}}$.

#2 (cont.) (c) As in part (b), it is enough to find a monic polynomial of degree 3

$$q(x) = x^3 + \alpha x^2 + \beta x + \gamma$$

which is orthogonal to every quadratic, linear, and constant polynomial on $[-1,1]$.
Since $\langle q, ax^2+bx+c\rangle = \bar{a}\langle q, x^2\rangle + \bar{b}\langle q, x\rangle + \bar{c}\langle q, 1\rangle$,
it suffices to find such a function q which is orthogonal to x^2, x, and 1 on $[-1,1]$.
This leads to the following three equations that the coefficients α, β, and γ of q must satisfy:

$$0 = \langle q, 1\rangle = \int_{-1}^{1} (x^3+\alpha x^2+\beta x+\gamma)\cdot 1\,dx = \left(\frac{x^4}{4} + \frac{\alpha x^3}{3} + \frac{\beta x^2}{2} + \gamma x\right)\Bigg|_{-1}^{1} = \frac{2\alpha}{3} + 2\gamma$$

$$0 = \langle q, x\rangle = \int_{-1}^{1} (x^3+\alpha x^2+\beta x+\gamma)x\,dx = \frac{x^5}{5} + \frac{\alpha x^4}{4} + \frac{\beta x^3}{3} + \frac{\gamma x^2}{2}\Bigg|_{-1}^{1} = \frac{2}{5} + \frac{2\beta}{3}$$

$$0 = \langle q, x^2\rangle = \int_{-1}^{1} (x^3+\alpha x^2+\beta x+\gamma)x^2\,dx = \frac{x^6}{6} + \frac{\alpha x^5}{5} + \frac{\beta x^4}{4} + \frac{\gamma x^3}{3}\Bigg|_{-1}^{1} = \frac{2\alpha}{5} + \frac{2\gamma}{3}.$$

From the second equation, $\beta = -3/5$. Multiply the third equation by (-3) and add the result to the first equation to get $-\frac{6\alpha}{5} + \frac{2\alpha}{3} = 0$, i.e. $\alpha = 0$, from which it easily follows that $\gamma = 0$, as well. Thus

$$\boxed{q(x) = x^3 - \frac{3}{5}x}.$$

Section 5.3, p. 118.

#3. Consider $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < l$, $0 < t < \infty$, with the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ for $t \geq 0$, and the initial conditions $u(x,0) = x$, $u_t(x,0) = 0$ for $0 \leq x < l$. Find the solution $u = u(x,t)$ explicitly in series form.

Separating variables via $u(x,t) = X(x)T(t)$ leads to

$$(\dagger) \begin{cases} X'' + \lambda X = 0, & X(0) = X'(l) = 0, \\ T'' + \lambda c^2 T = 0, & T'(0) = 0. \end{cases}$$

Consider the linear operator $L = \frac{-d^2}{dx^2}$ on the subspace $V = \{f \in C^2[0,l] : f(0) = f'(l) = 0\}$ of $C[0,l]$. If φ and ψ belong to V then

$$\left[\varphi(x)\overline{\psi'(x)} - \varphi'(x)\overline{\psi(x)}\right]\Big|_0^l = \varphi(l)\underbrace{\overline{\psi'(l)}}_{0} - \underbrace{\varphi'(l)}_{0}\overline{\psi(l)} - \underbrace{\varphi(0)}_{0}\overline{\psi'(0)} + \varphi'(0)\underbrace{\overline{\psi(0)}}_{0} = 0,$$

so $L = \frac{-d^2}{dx^2}$ is hermitian on V. Consequently all eigenvalues λ in $(\dagger)$ are real. Furthermore if $\varphi \in V$ is a real function then

$$\varphi(x)\varphi'(x)\Big|_0^l = \varphi(l)\underbrace{\varphi'(l)}_{0} - \underbrace{\varphi(0)}_{0}\varphi'(0) = 0,$$

so by exercise #13 of Sec. 5.3, all the eigenvalues λ in $(\dagger)$ satisfy $\lambda \geq 0$.

Therefore, we need only consider the following two cases.

<u>Case $\lambda = 0$.</u> Then $X(x) = c_1 x + c_2$ is the general solution to the ODE $X'' = 0$. $0 = X(0)$ implies $0 = c_2$ and $0 = X'(l)$ implies $0 = c_1$. Hence $\lambda = 0$ is not an eigenvalue in $(\dagger)$.

<u>Case $\lambda > 0$ (say $\lambda = \beta^2 > 0$).</u> Then $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$ is the general solution to the ODE $X'' + \lambda X = X'' + \beta^2 X = 0$.
$0 = X(0)$ implies $0 = c_1$ and $0 = X'(l)$ implies $0 = \beta c_2 \cos(\beta l)$. Thus

#3 (cont.) $\beta \neq 0$ and $\frac{c}{2} \neq 0$ (needed for a nontrivial solution) imply $\beta = \beta_n = \frac{(n+\frac{1}{2})\pi}{l}$ $(n = 0, 1, 2, \ldots)$. Therefore the eigenvalues and eigenfunctions are, respectively, $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2$ and $X_n(x) = \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$ where $n = 0, 1, 2, \ldots$. Substituting in the second equation of (†) we have

$$T_n''(t) + \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 c^2 T_n(t) = 0, \quad T_n'(0) = 0$$

and consequently $T_n(t) = \cos\left(\frac{(n+\frac{1}{2})\pi c t}{l}\right)$. A formal solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)\cos\left(\frac{(n+\frac{1}{2})\pi c t}{l}\right),$$

where the arbitrary constants $A_0, A_1, \ldots$ are chosen so that

$$x = u(x,0) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right) \quad \text{for all } 0 \le x < l.$$

Thus, choose the A_n's to be the Fourier coefficients of x with respect to the orthogonal set $\left\{\sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)\right\}_{n=0}^{\infty}$ on $(0, l)$:

$$A_n = \frac{\left\langle x, \sin\left(\frac{(n+\frac{1}{2})\pi(\cdot)}{l}\right)\right\rangle}{\left\langle \sin\left(\frac{(n+\frac{1}{2})\pi\cdot}{l}\right), \sin\left(\frac{(n+\frac{1}{2})\pi\cdot}{l}\right)\right\rangle} = \frac{\int_0^l x \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)dx}{\int_0^l \sin^2\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)dx} = \frac{2}{l}\int_0^l \underbrace{x}_{u} \underbrace{\sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)dx}_{dv}$$

$$= \cancel{\left(\frac{2}{l}\right)x\left(\frac{-l}{(n+\frac{1}{2})\pi}\right)\cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)\Bigg|_0^l}^{\,0} - \frac{2}{l}\int_0^l \frac{-l}{(n+\frac{1}{2})\pi}\cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)dx$$

$$= \frac{2}{(n+\frac{1}{2})\pi}\left[\frac{l}{(n+\frac{1}{2})\pi}\sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)\right]\Bigg|_0^l = \frac{2l\sin((n+\frac{1}{2})\pi)}{(n+\frac{1}{2})^2\pi^2} = \frac{2l(-1)^n}{(n+\frac{1}{2})^2\pi^2}.$$

Sec. 5.3, p. 118

#3 (cont.) Therefore

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2l(-1)^n}{(n+\frac{1}{2})^2\pi^2} \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right) \cos\left(\frac{(n+\frac{1}{2})\pi ct}{l}\right)$$

$$\boxed{u(x,t) = \frac{8l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{(2n+1)\pi x}{2l}\right) \cos\left(\frac{(2n+1)\pi ct}{2l}\right)}\ .$$

Sec. 5.3, pp. 118–120

#6 Find the complex eigenvalues of the first-derivative operator d/dx subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0,1)$?

Eigenvalue Problem: $\begin{cases} X'(x) = \lambda X(x) & \text{for} \quad 0 < x < 1 \\ X(0) = X(1) \end{cases}$

The general solution to the D.E. is $X(x) = c_1 e^{\lambda x}$. We want to find those numbers λ such that $X(0) = X(1)$. That is,

$$c_1 = c_1 e^0 = c_1 e^{\lambda} \quad \Rightarrow \quad \underbrace{c_1}_{\text{nonzero}}(1 - e^{\lambda}) = 0 .$$

Therefore $e^{\lambda} = 1$. The complex solutions λ to this equation are infinite in number and are of the form

$$(*) \qquad \boxed{\lambda_n = i2n\pi , \quad (n = 0, \pm 1, \pm 2, \dots) .}$$

To see this, note first that each number of the form $(*)$ __is__ a solution to $e^{\lambda} = 1$ due to Euler's identity:

$$e^{\lambda_n} = e^{i2n\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1 + 0i = 1 .$$

On the other hand, suppose $\lambda = a + bi$ (a and b real) is a solution to $e^{\lambda} = 1$. Then $e^{a+bi} = 1$ implies

$$e^a = \sqrt{e^{2a}} = \sqrt{e^{a+bi} \cdot e^{a-bi}} = \sqrt{e^{a+bi} \cdot \overline{e^{a+bi}}} = \sqrt{1 \cdot 1} = 1 .$$

But $a = 0$ is the only real solution to $e^a = 1$ (e^x is strictly increasing

Sec. 5.3, pp. 118-120

#6 (cont.) on $-\infty < x < \infty$). Consequently $\lambda = a+bi = 0+bi = bi$. Therefore $e^{\lambda} = 1$ becomes

$$\cos(b) + i\sin(b) = e^{ib} = 1 = 1 + 0i .$$

Equating real and imaginary parts gives $\cos(b) = 1$ and $\sin(b) = 0$. The only real solutions b to these two relations have the form $b = b_n = 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$ Thus $\lambda_n = ib_n = i2n\pi$ ($n = 0, \pm 1, \pm 2, \ldots$) are the only solutions possible for (*).

The eigenfunction X_n corresponding to $\lambda_n = i2n\pi$ are (up to a constant multiple)

$$X_n(x) = e^{\lambda_n x} = e^{i2\pi n x} = \cos(2\pi n x) + i\sin(2\pi n x).$$

If $m \neq n$ then X_n and X_m are orthogonal on $(0,1)$:

$$\int_0^1 X_n(x)\overline{X_m(x)}\,dx = \int_0^1 e^{i2\pi n x} \cdot e^{-i2\pi m x}\,dx$$

$$= \int_0^1 e^{i2\pi(n-m)x}\,dx$$

$$= \left. \frac{e^{i2\pi(n-m)x}}{i2\pi(n-m)} \right|_{x=0}^{x=1} \qquad (\text{since } m \neq n)$$

$$= \frac{1}{i2\pi(n-m)}\left[e^{i2\pi(n-m)} - 1\right] \overset{\text{by } (*)}{=} 0 .$$

Sec. 5.3, p. 119

#7. Show by direct integration that the eigenfunctions associated with the Robin boundary conditions, namely,

$$\varphi_n(x) = \cos(\beta_n x) + \frac{a_0}{\beta_n}\sin(\beta_n x) \qquad \text{where } \lambda_n = \beta_n^2,$$

are mutually orthogonal on $0 \le x \le \ell$, where β_n are the positive roots of

$$(4.3.8) \qquad (\beta^2 - a_0 a_\ell)\tan(\beta\ell) = (a_0 + a_\ell)\beta.$$

Recall that the sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ are eigen- tions to the eigenvalue problems

$$(\dagger) \quad \begin{cases} -\varphi_n''(x) = \lambda_n \varphi_n(x) & \text{for } 0 < x < \ell \\ \varphi_n'(0) - a_0\varphi_n(0) = 0 = \varphi_n'(\ell) + a_\ell \varphi_n(\ell) \end{cases}$$

$(n = 1, 2, 3, \ldots)$. Integrating by parts twice and using $(\dagger)$ gives

$$\lambda_n \int_0^\ell \varphi_n(x)\varphi_m(x)\,dx = \int_0^\ell \underbrace{\varphi_m(x)}_{v}\underbrace{\left[-\varphi_n''(x)\right]dx}_{dv}$$

$$= -\varphi_n'(x)\varphi_m(x)\Big|_0^\ell + \int_0^\ell \underbrace{\varphi_m'(x)}_{v}\underbrace{\varphi_n'(x)dx}_{dv}$$

$$= \left[\varphi_n(x)\varphi_m'(x) - \varphi_n'(x)\varphi_m(x)\right]\Big|_0^\ell + \int_0^\ell \varphi_n(x)\left[-\varphi_m''(x)\right]dx$$

$$= \left[\cancel{\varphi_n(\ell)(-a_\ell\varphi_m(\ell)) + (a_\ell\varphi_n(\ell))\varphi_m(\ell)} - \cancel{\varphi_n(0)(a_0\varphi_m(0)) + (a_0\varphi_n(0))\varphi_m(0)}\right]$$

$$+ \lambda_m \int_0^\ell \varphi_n(x)\varphi_m(x)\,dx.$$

Sec. 5.3

#7, p. 119 (cont.) Rearranging, we have

$$(\lambda_n - \lambda_m)\int_0^{\ell} \varphi_n(x)\varphi_m(x)\,dx = 0 .$$

If n and m are different then $\lambda_n \neq \lambda_m$, and it follows that

$$\int_0^{\ell} \varphi_n(x)\varphi_m(x)\,dx = 0;$$

that is, φ_n and φ_m are orthogonal on $(0,\ell)$.

#8, p. 119. Show directly that $(X_1 X_2' - X_1' X_2)\Big|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.

Suppose $X = X_1(x)$ and $X = X_2(x)$ both satisfy

$$X'(a) - \alpha X(a) = 0 = X'(b) + \beta X(b) .$$

Then

$$(X_1 X_2' - X_1' X_2)\Big|_a^b = X_1(b)X_2'(b) - X_1'(b)X_2(b) - X_1(a)X_2'(a) + X_1'(a)X_2(a)$$

$$= \cancel{X_1(b)(-\beta X_2(b))} + \cancel{\beta X_1(b) X_2(b)} - \cancel{X_1(a)(\alpha X_2(a))} + \cancel{\alpha X_1(a) X_2(a)}$$

$$= 0 .$$

Sec. 5.3

#11, p. 120. (a) Show that the condition $f(x)f'(x)\Big|_a^b \le 0$ is valid for any (differentiable) function f that satisfies Dirichlet, Neumann, or periodic boundary conditions

(b) Show that it is also valid for Robin boundary conditions provided that the constants a_0 and a_ℓ are positive.

(a) (Dirichlet) Suppose $f(a) = f(b) = 0$. Then

$$f(x)f'(x)\Big|_a^b = \underbrace{f(b)}_{0}f'(b) - \underbrace{f(a)}_{0}f'(a) = 0.$$

(Neumann) Suppose $f'(a) = f'(b) = 0$. Then

$$f(x)f'(x)\Big|_a^b = f(b)\underbrace{f'(b)}_{0} - f(a)\underbrace{f'(a)}_{0} = 0.$$

(Periodic) Suppose $f(b) - f(a) = 0 = f'(b) - f'(a)$. Then

$$f(x)f'(x)\Big|_a^b = f(b)\underbrace{f'(b)}_{\substack{\text{Replace}\\ \text{by } f'(a)}} - \underbrace{f(a)}_{\substack{\text{Replace}\\ \text{by } f(b)}}f'(a) = f(b)f'(a) - f(b)f'(a) = 0.$$

(b) (Robin) Suppose $f'(a) - a_0 f(a) = 0 = f'(b) + a_\ell f(b)$. Then

$$f(x)f'(x)\Big|_a^b = f(b)\underbrace{f'(b)}_{\substack{\text{Replace}\\ \text{by } -a_\ell f(b)}} - f(a)\underbrace{f'(\ )}_{\substack{\text{replace}\\ \text{by } a_0 f(a)}} = -\underbrace{a_\ell f^2(b)}_{\text{nonnegative}} - \underbrace{a_0 f^2(a)}_{\text{nonnegative}} \le 0.$$

Sec. 5.3, pp. 119–120.

#15. Use the same idea as in exercises 12 and 13 to show that none of the eigenvalues of the fourth-order differential operator $+d^4/dx^4$ with the boundary conditions $X(0) = X(\ell) = X''(0) = X''(\ell) = 0$ are negative.

Let λ be an eigenvalue of the problem above. Then there exists a (continuous) function $X = X(x)$, not identically zero, such that

$$
\begin{cases} X''''(x) = \lambda X(x) \quad \text{for } x \in (0,\ell), & (*) \\ X(0) = X(\ell) = X''(0) = X''(\ell) = 0 . & (**) \end{cases}
$$

Then integrating by parts twice and applying (**) and (*) gives

$$
\begin{aligned}
\lambda \int_0^\ell [X(x)]^2 dx &= \int_0^\ell X''''(x) X(x) dx \\
&= -\int_0^\ell X'''(x) X'(x) dx + X'''(x) X(x) \Big|_0^\ell \\
&= \int_0^\ell X''(x) X''(x) dx + \Big[X'''(x) X(x) - X''(x) X'(x) \Big] \Big|_0^\ell \\
&= \int_0^\ell [X''(x)]^2 dx \\
&\geq 0 .
\end{aligned}
$$

But $X \neq 0$ implies $\int_0^\ell [X(x)]^2 dx > 0$, so dividing through the inequality by this positive quantity yields $\lambda \geq 0$.