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#1. Consider the geometric series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$.

(a) Does it converge pointwise in the interval $-1 < x < 1$?

(b) Does it converge uniformly in the interval $-1 < x < 1$?

(c) Does it converge in the L^2 sense in the interval $-1 < x < 1$?

$$S_N(x) = \sum_{n=0}^{N} (-1)^n x^{2n} = \frac{1 - (-1)^{N+1} x^{2(N+1)}}{1 + x^2} . \qquad (*)$$

(We have used the fact that $\sum_{n=0}^{N} ar^n = \frac{a - ar^{N+1}}{1 - r}$ for a (finite) geometric series with first term a and common ratio $r \neq 1$.)

(a) If $-1 < x < 1$ then $\lim_{k \to \infty} |x|^k = 0$. Therefore

$$\lim_{N \to \infty} S_N(x) = \lim_{N \to \infty} \frac{1 + (-1)^N x^{2(N+1)}}{1 + x^2} = \frac{1}{1 + x^2} \quad \text{if } |x| < 1 .$$

$$\boxed{\text{Yes, the series converges (pointwise) in } (-1, 1), \text{ with sum } \frac{1}{1+x^2} .}$$

(b)
$$\underset{-1<x<1}{\text{l.u.b.}} \left| \frac{1}{1+x^2} - S_N(x) \right| \overset{\text{by } (*)}{=} \underset{-1<x<1}{\text{l.u.b.}} \left| \frac{(-1)^{N+1} x^{2(N+1)}}{1+x^2} \right| = \frac{1}{1+1^2} = \frac{1}{2} .$$

Therefore
$$\lim_{N \to \infty} \underset{-1<x<1}{\text{l.u.b.}} \left| \frac{1}{1+x^2} - S_N(x) \right| = \lim_{N \to \infty} \frac{1}{2} = \frac{1}{2} \neq 0,$$

so $\boxed{\text{the series does not converge uniformly in } -1 < x < 1 .}$

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#1 (c)

$$\left\| \frac{1}{1+x^2} - S_N(x) \right\|_{L^2}^2 = \int_{-1}^{1} \left| \frac{1}{1+x^2} - S_N(x) \right|^2 dx$$

$$\overset{\text{by } (*)}{=} \int_{-1}^{1} \left| \frac{1}{1+x^2} - \frac{1+(-1)^N x^{2(N+1)}}{1+x^2} \right|^2 dx$$

$$= \int_{-1}^{1} \frac{x^{4(N+1)}}{(1+x^2)^2} dx$$

$$= 2\int_{0}^{1} \frac{x^{4(N+1)}}{(1+x^2)^2} dx$$

$$\leq 2\int_{0}^{1} x^{4(N+1)} dx$$

$$= \left. \frac{2x^{4(N+1)+1}}{4(N+1)+1} \right|_0^1 = \frac{2}{4N+5}$$

Therefore $\lim_{N\to\infty} \left\| \frac{1}{1+x^2} - S_N(x) \right\|_{L^2} \leq \lim_{N\to\infty} \sqrt{\frac{2}{4N+5}} = 0,$

so the series converges in the L^2-sense to the function $\frac{1}{1+x^2}$ in the interval $(-1,1)$.

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#4. Let

$$g_n(x) = \begin{cases} 1 & \text{if} \quad x \in \left[\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}\right) \quad \text{and } n \text{ is odd,} \\ 1 & \text{if} \quad x \in \left[\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}\right) \quad \text{and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $g_n \to 0$ in the L^2-sense but that g_n does not tend to zero in the pointwise sense.

Since g_n is 1 on an interval of length $\frac{2}{n^2}$ and zero elsewhere,

$$\int_{-\infty}^{\infty} |g_n(x)|^2 dx = 1 \cdot \frac{2}{n^2} = \frac{2}{n^2} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thus $g_n \to 0$ in the L^2-sense. But $g_{2n}\left(\frac{3}{4}\right) = g_{2n+1}\left(\frac{1}{4}\right) = 1$ for all $n = 1, 2, 3, \ldots$ so $\{g_n(x)\}_{n=1}^{\infty}$ fails to converge to 0 if $x = \frac{3}{4}$ or $\frac{1}{4}$.

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#5. Let $\varphi(x) = \begin{cases} 0 & \text{if } 0<x<1, \\ 1 & \text{if } 1<x<3. \end{cases}$

(a) Find the first four nonzero terms of its Fourier cosine series explicitly.

(b) For each $x \in [0,3]$, what is the sum of this series?

(c) Does (the Fourier cosine series of φ) converge to φ in the L^2-sense? Why?

(d) Put $x=0$ to find the sum

$$1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+\cdots.$$

(a) $\varphi_n(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$ where $a_n = \frac{(\varphi, \cos(n\pi x/3))}{(\cos(n\pi x/3), \cos(n\pi x/3))} = \frac{2}{3}\int_0^3 \varphi(x)\cos\left(\frac{n\pi x}{3}\right)dx$

I.e. $a_n = \frac{2}{3}\int_1^3 1\cdot\cos\left(\frac{n\pi x}{3}\right)dx = \begin{cases} \frac{2}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_1^3 & \text{if } n\neq 0, \\ \frac{4}{3} & \text{if } n=0. \end{cases}$

Therefore $\boxed{\varphi(x) \sim \frac{4}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n}\cos\left(\frac{n\pi x}{3}\right)}$

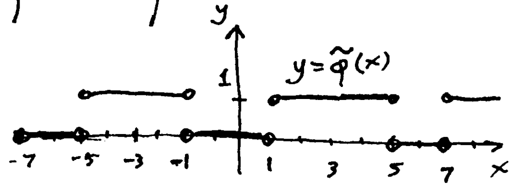
n	$\sin(n\pi/3)$
1	$\sqrt{3}/2$
2	$\sqrt{3}/2$
3	0
4	$-\sqrt{3}/2$
5	$-\sqrt{3}/2$
6	0
etc.	

Explicitly,

$$\boxed{\varphi(x) \sim \frac{4}{3} - \frac{\sqrt{3}}{\pi}\left[\cos\left(\frac{\pi x}{3}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{3}\right) - \frac{1}{4}\cos\left(\frac{4\pi x}{3}\right) - \frac{1}{5}\cos\left(\frac{5\pi x}{3}\right) + \cdots\right]}$$

(b) Because φ and φ' are piecewise continuous on $[0,3]$, by Theorem 4(ii), p. 125, the Fourier cosine series of φ converges at every $x \in (-\infty, \infty)$ to

(*) $$\frac{1}{2}\left[\tilde\varphi(x^+) + \tilde\varphi(x^-)\right]$$



where $\tilde\varphi$ is the even 6-periodic extension of φ. For $x \in [0,3]$ the sum (*) is

$$\boxed{\frac{4}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)\cos\left(\frac{n\pi x}{3}\right)}{n} = \begin{cases} 0 & \text{if } x \in [0,1), \\ \frac{1}{2} & \text{if } x=1, \\ 1 & \text{if } x \in (1,3]. \end{cases}}$$

#5(c) (cont.) Because $\int_0^3 |\varphi(x)|^2 dx = 2 < \infty$ and $\left\{ \cos\left(\frac{n\pi x}{3}\right) \right\}_{n=0}^{\infty}$ is the set of all eigenfunctions coming from the symmetric eigenvalue problem

$$-X''(x) = \lambda X(x) \quad \text{for } x \in (0,3) \text{ with } X'(0) = X'(3) = 0,$$

Theorem 3, p. 124, guarantees that $\boxed{\text{the Fourier cosine series of } \varphi \text{ converges in the } L^2\text{-sense to } \varphi \text{ on } [0,3].}$

(d) By parts (a) and (b)

$$0 = \varphi(0) = \frac{4}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n} \frac{1}{\cos(0)}$$

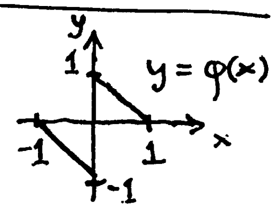
and thus

$$\frac{2\pi}{3} = \sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n} = \frac{\sqrt{3}}{2}\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots\right]$$

or

$$\boxed{\frac{4\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots}$$

#7, p. 130. Let $\varphi(x) = \begin{cases} 1 - x & \text{for } 0 < x < 1, \\ -(1+x) & \text{for } -1 < x < 0. \end{cases}$



(a) Find the full Fourier series of φ in the interval $(-1,1)$.

(b) Find the first three nonzero terms explicitly.

(c) Does it converge in the mean square sense? Why?

(d) Does it converge pointwise? Why?

(e) Does it converge uniformly? Why?

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#7, p. 130 (cont.)

(a) Since φ is an odd function, $a_n = 0$ for all $n \geq 0$, and

$$b_n = \frac{\langle \varphi, \sin(n\pi\,\cdot)\rangle}{\langle \sin(n\pi\,\cdot), \sin(n\pi\,\cdot)\rangle} = \frac{\int_{-1}^{1} \overbrace{\varphi(x)}^{\text{odd}} \overbrace{\sin(n\pi x)}^{\text{odd}}\,dx}{\underbrace{\int_{-1}^{1} \sin^2(n\pi x)\,dx}_{1}}$$

$$= \int_{-1}^{1} \overbrace{\varphi(x)\sin(n\pi x)}^{\text{even}}\,dx = 2\int_0^1 \varphi(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 \overbrace{(1-x)}^{v}\overbrace{\sin(n\pi x)\,dx}^{dv} = -\frac{2(1-x)\cos(n\pi x)}{n\pi}\Bigg|_0^1 - \frac{2}{n\pi}\cancel{\int_0^1 \cos(n\pi x)\,dx}^{\,0}$$

$$= \frac{2}{n\pi}.$$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)}$ is the full Fourier series of φ in $(-1,1)$.

(b) The full Fourier series of φ in $(-1,1)$ is, explicitly,

$$\boxed{\frac{2}{\pi}\left(\sin(\pi x) + \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) + \dots\right).}$$

(c) Since $\int_{-1}^{1} |\varphi(x)|^2\,dx \leq \int_{-1}^{1} 1\,dx = 2 < \infty$, Theorem 3 of Sec 5.4 guarantees that the full Fourier series of φ converges in the mean-square sense to φ on $(-1,1)$.

(d) Since φ and φ' are piecewise continuous on $-1 \leq x \leq 1$ (Note that $\varphi'(x) = -1$ if $0 < |x| < 1$.), it follows from Theorem 4(ii)

#7, p. 130 (cont.) that the full Fourier series of φ converges at every $x \in [-1,1]$ and

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} = \frac{\varphi(x^+)+\varphi(x^-)}{2}$$

$$= \begin{cases} \varphi(x) & \text{if } 0<|x|<1, \\ 0 & \text{if } x = 0, \pm 1. \end{cases}$$

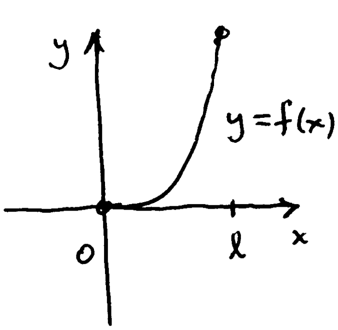
(e) The full Fourier series does not converge uniformly on $[-1,1]$ since the sum function (see above) is not continuous at 0, for instance. [See the Convergence Theorem, Appendix A.2, p. 389.]

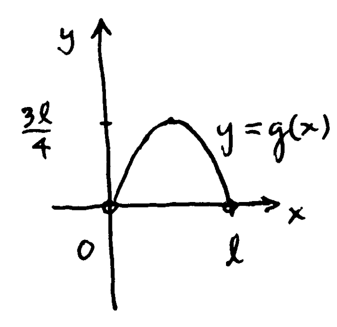
#8, p. 130. Consider the Fourier sine series of each of the following functions. In this exercise, do not compute the coefficients, but use the general convergence theorems (Theorems 2, 3, and 4) to discuss the convergence of each of the series in the pointwise, uniform, and L^2 senses.

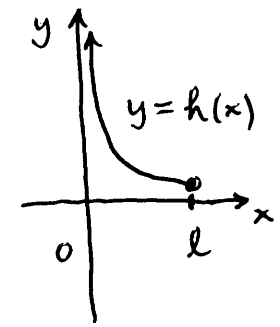
(a) $f(x) = x^3$ on $(0,\ell)$.

(b) $g(x) = \ell x - x^2$ on $(0,\ell)$.

(c) $h(x) = x^{-2}$ on $(0,\ell)$.







#8, p. 130 (cont.) (a) Since f and f' are continuous on $[0,\ell]$, Theorem 4 implies that the Fourier sine series of f converges pointwise at every $x \in [0,\ell]$, and the sum function is

$$\begin{cases} x^3 & \text{if } 0 \le x < \ell, \\ 0 & \text{if } x = 0. \end{cases}$$

The Fourier sine series of f fails to converge uniformly on $[0,\ell]$ since the sum function (see above) is discontinuous at $x = \ell$ [cf. the Convergence Theorem, Appendix A.2, p. 389]. Because

$$\int_0^\ell |f(x)|^2 dx = \int_0^\ell x^6 dx = \ell^7/7 < \infty,$$

Theorem 3 implies that the Fourier sine series of f converges to f in the mean-square sense in $(0,\ell)$.

(b) Note that the functions $X_n(x) = \sin(n\pi x/\ell)$, where $n = 1,2,3,\ldots$, are the (complete) set of eigenfunctions for the eigenvalue problem with (symmetric!) Dirichlet B.C.'s:

$$X''(x) + \lambda X(x), \quad X(0) = X(\ell) = 0.$$

Also, observe that $g \in C^2([0,\ell])$ and g satisfies the B.C.'s above. Consequently, the Fourier sine series of g converges uniformly to g on $[0,\ell]$ by Theorem 2. By exercise 2, Sec. 5.4 (worked in lecture), the Fourier sine series of g converges pointwise and in the mean-square sense to g on $[0,\ell]$.

(c) Since h is not integrable on $(0,\ell)$, the

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#8, p. 130 (cont.) Fourier sine coefficients of h are not even defined!
(The improper integrals

$$\int_0^{\ell} \frac{\sin(n\pi x/\ell)}{x^2}\,dx$$

are not convergent.) The convergence questions are meaningless for the Fourier sine series of $h(x) = x^{-2}$ on $(0,\ell)$.

#12, p. 131. Start with the Fourier sine series of $f(x) = x$ on the interval $(0,\ell)$. Apply Parseval's identity. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The Fourier sine series for $f(x) = x$ on $(0,\ell)$ is

$$x \sim \sum_{m=1}^{\infty} \frac{2\ell(-1)^{m+1}}{\pi m} \sin\left(\frac{m\pi x}{\ell}\right).$$

(See example 3 of Sec. 5.1.) By Theorem 3, the Fourier sine series of f converges in the mean-square sense to f on $(0,\ell)$. Consequently, the Parseval identity (19) on p. 128 implies

$$\sum_{m=1}^{\infty} \left(\frac{2\ell(-1)^{m+1}}{\pi m}\right)^2 \underbrace{\int_0^{\ell} \sin^2\left(\frac{m\pi x}{\ell}\right)dx}_{\ell/2} = \underbrace{\int_0^{\ell} x^2\,dx}_{\ell^3/3}.$$

Hence

$$\frac{2\ell^3}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\ell^3}{3},$$ from which it follows that $\boxed{\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}}$.

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#13. Start with the Fourier cosine series of $f(x) = x^2$ on the interval $(0, \ell)$. Apply Parseval's identity. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

$$a_n = \frac{(f, \cos(n\pi x/\ell))}{(\cos(n\pi x/\ell), \cos(n\pi x/\ell))} = \frac{2}{\ell}\int_0^\ell x^2 \cos\left(\frac{n\pi x}{\ell}\right) dx \qquad \text{(Now integrate by parts twice)}$$

$$= \frac{2}{\ell}\left[\frac{x^2 \sin(n\pi x/\ell)}{(n\pi/\ell)} + \frac{2x\cos(n\pi x/\ell)}{(n\pi/\ell)^2} - \frac{2\sin(n\pi x/\ell)}{(n\pi/\ell)^3} \right]\Bigg|_0^\ell$$

$$= \frac{2}{\ell} \cdot \frac{2\ell\cos(n\pi)}{(n\pi/\ell)^2}$$

$$= \frac{4\ell^2(-1)^n}{n^2\pi^2} \qquad (n \neq 0)$$

$$a_0 = \frac{(f, 1)}{(1, 1)} = \frac{1}{\ell}\int_0^\ell x^2\,dx = \frac{1}{\ell}\cdot\frac{x^3}{3}\Bigg|_0^\ell = \frac{\ell^2}{3}$$

$$x^2 = f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) = \frac{\ell^2}{3} + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x/\ell)}{n^2}.$$

We now apply Parseval's identity (19), p. 128:

$$\sum_{n=0}^{\infty} a_n^2 \int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right) dx = \int_0^\ell [f(x)]^2\,dx$$

to obtain

$$\left(\frac{\ell^2}{3}\right)^2 \cdot \ell + \sum_{n=1}^{\infty} \left(\frac{4\ell^2(-1)^n}{n^2\pi^2}\right)^2 \cdot \frac{\ell}{2} = \frac{x^5}{5}\Bigg|_0^\ell$$

$$\frac{\cancel{\ell^5}}{9} + \frac{8\cancel{\ell^5}}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\cancel{\ell^5}}{5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{4\pi^4}{8\cdot 45} = \boxed{\frac{\pi^4}{90}}$$

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#15 Let $\varphi(x) \equiv 1$ for $0 < x < \pi$. Expand

$$1 = \sum_{n=0}^{\infty} B_n \cos\left[(n+\tfrac{1}{2})x\right].$$

(a) Find B_n.

(b) Let $-2\pi < x < 2\pi$. For which x does this series converge? For each such $x \in (-2\pi, 2\pi)$, what is the sum of the series?

(c) Apply Parseval's equality to this series. Use it to calculate the sum

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots .$$

(a)
$$B_n = \frac{\left(\varphi,\ \cos((n+\frac{1}{2})x)\right)}{\left(\cos((n+\frac{1}{2})x),\ \cos((n+\frac{1}{2})x)\right)} \qquad (n = 0, 1, 2, \dots)$$

$$= \frac{\int_0^{\pi} 1 \cdot \cos((n+\frac{1}{2})x)\,dx}{\int_0^{\pi} \cos^2((n+\frac{1}{2})x)\,dx}$$

$$= \frac{\frac{1}{n+\frac{1}{2}} \sin((n+\frac{1}{2})x)\Big|_0^{\pi}}{\frac{1}{2}\int_0^{\pi}\left[1 + \cos(2(N+\frac{1}{2})x)\right]dx}$$

$$= \frac{\frac{1}{n+\frac{1}{2}} \sin((n+\frac{1}{2})\pi)}{\frac{1}{2}x + \frac{1}{2}\frac{1}{(2N+1)}\sin((2N+1)x)\Big|_0^{\pi}}$$

$$= \frac{1}{n+\frac{1}{2}}(-1)^n \Big/ \pi/2 = \boxed{\frac{2(-1)^n}{(n+\frac{1}{2})\pi}}$$

n	$\sin((n+\frac{1}{2})\pi)$
0	$\sin \pi/2 = 1$
1	$\sin 3\pi/2 = -1$
2	$\sin 5\pi/2 = 1$
$\vdots$	
n	$(-1)^n$

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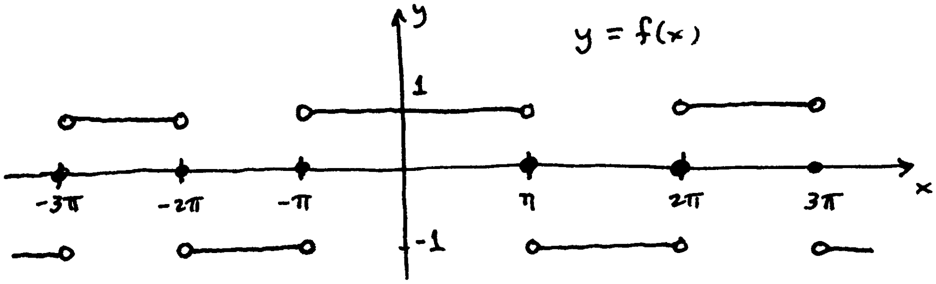
#15(b) Claim:

$$\boxed{\sum_{n=0}^{\infty} \frac{2(-1)^n}{(n+\frac{1}{2})\pi} \cos\left((n+\tfrac{1}{2})x\right) = \begin{cases} 1 & \text{if } |x| < \pi, \\ 0 & \text{if } x = \pm\pi, \pm 2\pi, \\ -1 & \text{if } \pi < |x| < 2\pi. \end{cases}}$$

To see this extend the function $\varphi(x) \equiv 1$ on $(0,\pi)$ to ψ on $[0, 2\pi)$ given by $\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi, \\ 0 & \text{if } x = \pi. \\ -1 & \text{if } \pi < x < 2\pi, \end{cases}$ and then let f be the even, 4π-periodic extension of ψ to the whole real line.



Since f and f' are piecewise continuous, Theorem 4^{∞}, p. 125, implies that the classical full Fourier series of f at x converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ for all $x \in (-\infty, \infty)$.

Since f is even and $\int_0^{2\pi} f(x)\,dx = 0$, we have $a_0 = 0$ and $b_n = 0$ for all $n = 1, 2, 3, \dots$ in the full Fourier series of f. Thus

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right] = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \overset{l=2\pi}{=} \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right)$$

where $a_n = \dfrac{(f, \cos(\frac{n}{2}x))}{(\cos(\frac{n}{2}x), \cos(\frac{n}{2}x))} = \dfrac{\int_{-2\pi}^{2\pi} f(x)\cos(\frac{nx}{2})\,dx}{\int_{-2\pi}^{2\pi} \cos^2(\frac{nx}{2})\,dx} = \dfrac{2\int_0^{2\pi} \psi(x)\cos(\frac{nx}{2})\,dx}{2\pi}.$

#15(b) (cont.) If $n \neq 0$ then

$$\int_0^{\pi} 1 \cdot \cos\left(\frac{nx}{2}\right)dx = \frac{2}{n}\sin\left(\frac{nx}{2}\right)\Bigg|_0^{\pi} = \frac{2}{n}\sin\left(\frac{n\pi}{2}\right)$$

and

$$\int_{\pi}^{2\pi} -1 \cdot \cos\left(\frac{nx}{2}\right)dx = \frac{-2}{n}\sin\left(\frac{nx}{2}\right)\Bigg|_{\pi}^{2\pi} = \frac{2}{n}\sin\left(\frac{n\pi}{2}\right).$$

Therefore

$$\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right)dx = \int_0^{\pi} 1 \cdot \cos\left(\frac{nx}{2}\right)dx + \int_{\pi}^{2\pi} -1 \cdot \cos\left(\frac{nx}{2}\right)dx = \frac{4}{n}\sin\left(\frac{n\pi}{2}\right),$$

and consequently,

$$a_n = \frac{2\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right)dx}{2\pi} = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Thus

$$\begin{cases} a_{2m} = 0, \\ a_{2m+1} = \dfrac{4(-1)^m}{(2m+1)\pi} = \dfrac{2(-1)^m}{(m+\frac{1}{2})\pi}, \end{cases}$$

n	$\sin(n\pi/2)$
1	1
2	0
3	−1
4	0
⋮	⋮
$2m$	0
$2m+1$	$(-1)^m$

where $m = 0, 1, 2, \ldots$. It follows that

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right)$$

$$= \sum_{m=0}^{\infty} \frac{2(-1)^m}{(m+\frac{1}{2})\pi}\cos\left((2m+1)\frac{x}{2}\right) = \sum_{m=0}^{\infty} \frac{2(-1)^m}{(m+\frac{1}{2})\pi}\cos\left((m+\tfrac{1}{2})x\right),$$

for all $x \in (-\infty, \infty)$. Comparison of this identity with the graph of f on the previous page makes clear the truth of the claim.

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#15(c) We apply Parseval's equality to the function f (in part (b) of this problem) on the interval $(-2\pi, 2\pi)$ with the orthogonal set of functions $\left\{\frac{1}{2}, \cos\left(\frac{nx}{2}\right), \sin\left(\frac{nx}{2}\right)\right\}_{n=1}^{\infty}$. Thus

$$|a_0|^2 \int_{-2\pi}^{2\pi} \left(\tfrac{1}{2}\right)^2 dx + \sum_{n=1}^{\infty}\left(|a_n|^2 \int_{-2\pi}^{2\pi} \cos^2\left(\tfrac{nx}{2}\right)dx + |b_n|^2 \int_{-2\pi}^{2\pi} \sin^2\left(\tfrac{nx}{2}\right)dx\right) = \int_{-2\pi}^{2\pi} |f(x)|^2 dx$$

$$\sum_{m=0}^{\infty} \left| \frac{2(-1)^m}{(m+\frac{1}{2})\pi} \right|^2 (2\pi) = \int_{-2\pi}^{2\pi} 1\, dx = 4\pi$$

$$\sum_{m=0}^{\infty} \left| \frac{4(-1)^m}{(2m+1)\pi} \right|^2 = 2$$

$$\frac{16}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = 2$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}}.$$

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#16, p.131. Let $\varphi(x) = |x|$ in $(-\pi, \pi)$. If we approximate it by a function of the form

$$(\dagger) \qquad f(x) = \frac{a_0}{2} + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x),$$

what choice of coefficients will minimize the L^2 error?

By Theorem 5, the L^2 error will be minimized when the coefficients are the Fourier coefficients of with respect to the orthogonal set $\{\frac{1}{2}, \cos(x), \sin(x), \cos(2x), \sin(2x)\}$. Thus, to minimize L^2 error,

$$a_0 = \frac{\langle \varphi, 1/2 \rangle}{\langle 1/2, 1/2 \rangle} = \frac{\int_{-\pi}^{\pi} \varphi(x)\frac{1}{2}\,dx}{\int_{-\pi}^{\pi} (\frac{1}{2})^2\,dx} = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$u = x$, $dV = \cos(nx)dx$

$$a_n = \frac{\langle \varphi, \cos(n\cdot) \rangle}{\langle \cos(n\cdot), \cos(n\cdot) \rangle} = \frac{\int_{-\pi}^{\pi} \varphi(x)\cos(nx)\,dx}{\int_{-\pi}^{\pi} \cos^2(nx)\,dx} = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$du = dx$, $V = \frac{\sin(nx)}{n}$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n}\right)\Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin(nx)\,dx = \cancel{\frac{2x\sin(nx)}{n\pi}}\Big|_0^{\pi} + \frac{2\cos(nx)}{\pi n^2}\Big|_0^{\pi}$$

$$= \frac{2(-1)^n - 2}{\pi n^2} = \begin{cases} 0 & \text{if } n = 2m \text{ is even,} \\ \frac{-4}{\pi n^2} & \text{if } n = 2m-1 \text{ is odd.} \end{cases}$$

$$b_n = \frac{\langle \varphi, \sin(n\cdot) \rangle}{\langle \sin(n\cdot), \sin(n\cdot) \rangle} = \frac{\int_{-\pi}^{\pi} \varphi(x)\sin(nx)\,dx}{\int_{-\pi}^{\pi} \sin^2(nx)\,dx} = 0 \quad \text{since } \varphi \text{ is even.}$$

Therefore $\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x)}$ is the L^2 error minimizing approximation $(\dagger)$.

Sec. 5.4, pp. 129–131.

#18. Consider a solution of the wave equation with $c = 1$ on $[0, \ell]$ with homogeneous Dirichlet or Neumann boundary conditions.

(a) Show that its energy $E = \frac{1}{2}\int_0^\ell (u_t^2 + u_x^2)\,dx$ is a constant.

(b) Let $E_n(t)$ be the energy of its nth harmonic (the nth term in the (series) expansion). Show that $E = \sum E_n$.

(a) See exercise #11, Sec. 4.3, parts (a) and (b).

(b) The solution $u = u(x,t)$ is given formally by either

$$(\dagger) \qquad u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi t}{\ell}\right) + B_n\sin\left(\frac{n\pi t}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right) \qquad ((9)\ \text{Sec. } 4.1)$$

or

$$(\dagger\dagger) \qquad u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi t}{\ell}\right) + B_n\sin\left(\frac{n\pi t}{\ell}\right)\right]\cos\left(\frac{n\pi x}{\ell}\right) \qquad ((7)\ \text{Sec. } 4.2)$$

depending on whether the BC's are (homogeneous) Dirichlet or Neumann. We do case $(\dagger)$, the other being completely analogous. The nth harmonic of $(\dagger)$ is

$$h_n(x,t) = \left[A_n\cos\left(\frac{n\pi t}{\ell}\right) + B_n\sin\left(\frac{n\pi t}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right).$$

Substituting the expressions

$$\frac{\partial h_n}{\partial t} = \frac{n\pi}{\ell}\left[-A_n\sin\left(\frac{n\pi t}{\ell}\right) + B_n\cos\left(\frac{n\pi t}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right),$$

$$\frac{\partial h_n}{\partial x} = \left[A_n\cos\left(\frac{n\pi t}{\ell}\right) + B_n\sin\left(\frac{n\pi t}{\ell}\right)\right]\left(\frac{n\pi}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right),$$

into $E_n(t) = \frac{1}{2}\int_0^\ell \left[\left(\frac{\partial h_n}{\partial t}\right)^2 + \left(\frac{\partial h_n}{\partial x}\right)^2\right]dx$ gives

Sec. 5.4

#18, p.131 (cont.)

Both these integrals are $\ell/2$.

$$(*) \quad E_n(t) = \frac{1}{2}\left(\frac{n\pi}{\ell}\right)^2\left[-A_n\sin\left(\frac{n\pi t}{\ell}\right)+B_n\cos\left(\frac{n\pi t}{\ell}\right)\right]^2\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)dx$$

$$+ \frac{1}{2}\left(\frac{n\pi}{\ell}\right)^2\left[A_n\cos\left(\frac{n\pi t}{\ell}\right)+B_n\sin\left(\frac{n\pi t}{\ell}\right)\right]^2\int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{\ell}{4}\left(\frac{n\pi}{\ell}\right)^2\Big\{A_n^2\sin^2\left(\frac{n\pi t}{\ell}\right) - \cancel{2A_nB_n\sin\left(\frac{n\pi t}{\ell}\right)\cos\left(\frac{n\pi t}{\ell}\right)} + B_n^2\cos^2\left(\frac{n\pi t}{\ell}\right)$$

$$+ A_n^2\cos^2\left(\frac{n\pi t}{\ell}\right) + \cancel{2A_nB_n\sin\left(\frac{n\pi t}{\ell}\right)\cos\left(\frac{n\pi t}{\ell}\right)} + B_n^2\sin^2\left(\frac{n\pi t}{\ell}\right)\Big\}$$

$$= \frac{(n\pi)^2}{4\ell}\left[A_n^2+B_n^2\right]$$

On the other hand, differentiating (†) term-by-term and setting $t=0$, we find

$$(**) \quad u_t(x,0) = \sum_{n=1}^\infty \frac{n\pi}{\ell}B_n\sin\left(\frac{n\pi x}{\ell}\right) \quad \text{and} \quad u_x(x,0) = \sum_{n=1}^\infty \frac{n\pi}{\ell}A_n\cos\left(\frac{n\pi x}{\ell}\right)$$

for $0\le x\le \ell$. Using part (a) and Parseval's identity [(19) p.128] yields

$$(***) \quad E(t) = E(0) = \frac{1}{2}\int_0^\ell \left(u_t^2(x,0) + u_x^2(x,0)\right)dx$$

(**) and Parseval   (**) and Parseval

$$= \frac{1}{2}\left(\sum_{n=1}^\infty \left|\frac{n\pi}{\ell}B_n\right|^2 \underbrace{\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)dx}_{\ell/2}\right) + \frac{1}{2}\left(\sum_{n=1}^\infty \left|\frac{n\pi}{\ell}A_n\right|^2 \underbrace{\int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right)dx}_{\ell/2}\right)$$

$$= \sum_{n=1}^\infty \frac{(n\pi)^2B_n^2}{4\ell} + \sum_{n=1}^\infty \frac{(n\pi)^2A_n^2}{4\ell}$$

$$= \sum_{n=1}^\infty \frac{(n\pi)^2}{4\ell}\left[A_n^2+B_n^2\right].$$

Comparing (*) and (***), we reach the desired conclusion: $E = \sum_{n=1}^\infty E_n$.