Sec. 6.1, pp. 154–155.

#1. Show that a function which is a power series in the complex variable $z = x + iy$ must satisfy the Cauchy-Riemann equations, and therefore Laplace's equation.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r$ and write

$f(z) = u(z) + i v(z) = u(x+iy) + iv(x+iy)$ where u and v are real-valued functions. (u is called the real part of f and v the imaginary part of f.) We must show that

$$(*) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(Cauchy-Riemann equations)}$$

at all points in the disk $x^2 + y^2 = |z|^2 < r^2$.

Note that

$$u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} a_n (x+iy)^n \quad \text{for} \quad x^2+y^2 < r^2.$$

Differentiating the convergent power series first with respect to x and second with respect to y gives

$$u_x + i v_x = \sum_{n=0}^{\infty} n a_n (x+iy)^{n-1}$$

and

$$u_y + i v_y = \sum_{n=0}^{\infty} i n a_n (x+iy)^{n-1} .$$

Multiplying the second equation by i and adding the result to the first equation gives

$$(u_x - v_y) + i(v_x + u_y) = \sum_{1}^{\infty} (na_n - na_n)(x+iy)^{n-1} = 0 \;(= 0 + 0i)$$

for all $x^2+y^2 < r^2$. Therefore $u_x - v_y = v_x + u_y = 0$, i.e. $(*)$ holds, in the disk $x^2+y^2<r^2$.

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#2. Find the solutions that depend only on r of the equation

$$(*) \qquad u_{xx} + u_{yy} + u_{zz} = k^2 u$$

where k is a positive constant.

Let $v = ru$. Then $u = \frac{v}{r} = r^{-1}v$ so

$$(**) \qquad \frac{\partial u}{\partial r} = -r^{-2}v + r^{-1}\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2} = 2r^{-3}v - 2r^{-2}\frac{\partial v}{\partial r} + r^{-1}\frac{\partial^2 v}{\partial r^2}.$$

If u is a function of r only, then via (6) p.153, (*) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = k^2 u,$$

and substituting from (**) we have

$$\cancel{\frac{2v}{r^3}} - \cancel{\frac{2v_r}{r^2}} + \frac{v_{rr}}{r} + \frac{2}{r}\left(-\cancel{\frac{v}{r^2}} + \cancel{\frac{v_r}{r}}\right) = \frac{k^2 v}{r}$$

$$v_{rr} - k^2 v = 0.$$

Therefore $v(r) = c_1 e^{-kr} + c_2 e^{kr}$, and hence

$$\boxed{u(r) = r^{-1}v(r) = \frac{c_1 e^{-kr} + c_2 e^{kr}}{r}}.$$

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#4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $a < r < b$ $(a > 0)$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants.

Because the B.C.'s are radially symmetric, we seek a solution u which is radially symmetric, i.e. u is a function of r only. By (6), p.153 the PDE then becomes

$$(*) \qquad u_{rr} + \frac{2}{r} u_r = 0 \qquad \text{in} \qquad a < r < b,$$

and the B.C.'s are $u(a) = A$ and $u(b) = B$. Rewriting $(*)$ as

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

we see that $r^2 u_r = c_1$ (upon integrating w.r.t. r once)

or $\dfrac{\partial u}{\partial r} = c_1 r^{-2}$.

Integrating w.r.t. r again yields

$$u = -\frac{c_1}{r} + c_2 .$$

Applying the B.C.'s we have

$$A = u(a) = -\frac{c_1}{a} + c_2 ,$$

$$B = u(b) = \frac{-c_1}{b} + c_2 .$$

Subtracting equations gives

$$A - B = c_1\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{c_1(a-b)}{ab} \qquad \text{or} \qquad c_1 = \frac{ab(A-B)}{a-b}.$$

Consequently $c_2 = A + \dfrac{c_1}{a} = A + \dfrac{b(A-B)}{a-b} = \dfrac{aA - bB}{a-b}$.

Thus $u = c_2 - \dfrac{c_1}{r} = \dfrac{c_2 r - c_1}{r} = \boxed{\dfrac{(aA-bB)r - ab(A-B)}{(a-b)r}}$.

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#6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $(0<)\, a < \sqrt{x^2+y^2} < b$ with $u = u(x,y)$ vanishing on both parts of the boundary $\sqrt{x^2+y^2} = a$ and $\sqrt{x^2+y^2} = b$.

Since the PDE and the boundary conditions are invariant under rotations, we seek (and expect) a solution $u = u(r)$ that is a radial function, independent of the polar angle θ. Expressing the problem in polar coordinates $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \cancel{\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}}^{\,0} = 1 \\ u(a) = u(a,\theta) = 0 \quad \text{and} \quad u(b) = u(b,\theta) = 0 . \end{cases}$$

Thus $\frac{1}{r}(ru')' = 1$ (where $'$ denotes differentiation with respect to r) and hence

$$(ru')' = r$$

$$\Rightarrow \quad ru' = \frac{r^2}{2} + c_1$$

$$\Rightarrow \quad u' = \frac{r}{2} + \frac{c_1}{r}$$

$$\Rightarrow \quad u = \frac{r^2}{4} + c_1 \log(r) + c_2 .$$

Applying the boundary conditions we have

$$\left.\begin{aligned} 0 &= u(a) = \frac{a^2}{4} + c_1 \log(a) + c_2 \\ 0 &= u(b) = \frac{b^2}{4} + c_1 \log(b) + c_2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} c_1 &= \frac{a^2-b^2}{4\log(b/a)} \\ c_2 &= -\frac{a^2}{4} - \frac{(a^2-b^2)\log(a)}{4\log(b/a)} \end{aligned}$$

Therefore, upon substituting these expressions for c_1 and c_2 and simplifying:

$$\boxed{u = \frac{r^2 - a^2}{4} + \frac{(a^2-b^2)\log(r/a)}{4\log(b/a)}} .$$

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#9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at $100°C$. Its outer boundary satisfies $\frac{\partial u}{\partial r} = -\gamma < 0$, where γ is a constant.

(a) Find the temperature (distribution function).

(b) What are the hottest and coldest temperatures?

(c) Can you choose γ so that the temperature on its outer boundary is $20°C$?

The temperature distribution function u satisfies Laplaces equation $\nabla^2 u = 0$ in the region $1 < r < 2$. Since the B.C.'s are radially symmetric, we assume a like solution, i.e. $u = u(r)$. Using (6) p. 153 we see that $u = u(r)$ satisfies

$$\begin{cases} u_{rr} + \frac{2}{r} u_r = 0 & \text{for } 1 < r < 2, \\ u(1) = 100, \\ u_r(2) = -\gamma. \end{cases}$$

Rewriting the PDE as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0,$$

we see that $r^2 u_r = c_1$ or $u_r = \frac{c_1}{r^2}$, and hence $u = -\frac{c_1}{r} + c_2$. Applying the B.C.'s gives

$$\left.\begin{aligned} 100 = u(1) &= -c_1 + c_2 \\ -\gamma = u_r(2) &= \frac{c_1}{4} \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 100 - 4\gamma, \\ c_1 &= -4\gamma. \end{aligned}$$

Thus (a) $\boxed{u(r) = \frac{4\gamma}{r} + 100 - 4\gamma}$ for $1 < r < 2$.

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#9. (cont.) (b) From (a), $u = u(r)$ is clearly a decreasing function of r on $1 \le r \le 2$. Thus

$$\boxed{\begin{aligned} u_{max} &= u(1) = 100^\circ C, \\ u_{min} &= u(2) = 2\gamma + 100 - 4\gamma = 100 - 2\gamma. \end{aligned}}$$

(c) The desired condition is

$$20 = u(2) = 100 - 2\gamma.$$

Therefore if $\boxed{\gamma = 40}$ the temperature on the outer boundary is $20^\circ C$.

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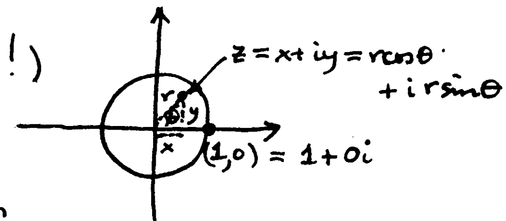
#12. Check the validity of the maximum principle for the harmonic function

$$u(x,y) = \frac{1-(x^2+y^2)}{1-2x+x^2+y^2}$$

in the disk $\overline{D} = \{(x,y): x^2+y^2 \le 1\}$. Explain.

Note that in the open disk $|z| = \sqrt{x^2+y^2} < 1$ in the complex plane, we have

$$(*) \qquad f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \quad \text{(geometric series!)}$$



By exercise # 1, the real and imaginary parts of f are harmonic functions in the disk $|z| = |x+iy| = \sqrt{x^2+y^2} < 1$. But

$$f(z) = \frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})} = \frac{1-r\cos\theta+ir\sin\theta}{1-(z+\bar{z})+|z|^2} = \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2},$$

$$(**) \qquad f(z) = \frac{\overbrace{1-x}^{\text{real part of } f}}{1-2x+x^2+y^2} + i\,\frac{\overbrace{y}^{\text{imaginary part of } f}}{1-2x+x^2+y^2} = v(x,y) + i\,w(x,y).$$

Therefore $v(x,y) = \dfrac{1-x}{1-2x+x^2+y^2}$ is harmonic in the open disk $x^2+y^2 < 1$.

But $$u(x,y) = \frac{1-(x^2+y^2)}{(x-1)^2+y^2} = \frac{1-[(x-1)^2-2x-1+y^2]}{(x-1)^2+y^2} = \frac{2-2x-[(x-1)^2+y^2]}{(x-1)^2+y^2}$$

$$(***) \qquad u(x,y) = 2\left(\frac{1-x}{(x-1)^2+y^2}\right) - 1 = 2v(x,y) - 1.$$

Therefore u is the linear combination of the two harmonic functions v and 1 in the open unit disk so u is harmonic there as well.

#12. (cont.) Note that for $|z| = |x+iy| < 1$, using (*), (**), and (***) we have

$$\frac{1+z}{1-z} = \frac{2}{1-z} - 1 = 2f(z) - 1 = 2\left[v(x,y) + iw(x,y)\right] - 1$$

$$= \left[2v(x,y) - 1\right] + i2w(x,y)$$

$$= \underbrace{u(x,y)}_{\substack{\text{real part} \\ \text{of } \frac{1+z}{1-z}}} + \underbrace{i2w(x,y)}_{\substack{\text{imaginary part} \\ \text{of } \frac{1+z}{1-z}}} .$$

Because $\frac{1+z}{1-z}$ has a singularity at $z = 1$ (i.e. $x+iy = 1+0i$), it follows that $u = u(x,y)$ has a singularity at the boundary point $(x,y) = (1,0)$ of the unit disk. We compute:

$$\lim_{x \to 1^-} u(x,0) = \lim_{x \to 1^-} \frac{1-x^2}{(x-1)^2} \overset{\text{l'Hôpital}}{=} \lim_{x \to 1^-} \frac{-2x}{2(x-1)} = +\infty .$$

Therefore $u = u(x,y) = \dfrac{1-(x^2+y^2)}{(1-x)^2+y^2}$ "achieves" its maximum "value" $+\infty$ on the closed unit disk $\overline{D} = \{(x,y) : x^2+y^2 \le 1\}$ at the boundary point $(1,0)$ of $\overline{D}$.