Sec. 6.3, pp. 163–164.

#1 Let u be a harmonic function in the disk $D = \{(r;\theta): 0 \le r < 2, -\pi \le \theta < \pi\}$ and let $u(2;\theta) = 3\sin(2\theta)+1$ for $-\pi \le \theta < \pi$. Without finding the solution, answer the following questions.

(a) Find the maximum value of u in $\bar{D}$.

(b) Calculate the value of u at the origin.

(a) $$\max_{(r;\theta)\in\bar{D}} u(r;\theta) \overset{\text{(Max Principle)}}{=} \max_{(r;\theta)\in\partial D} u(r;\theta) = \max_{-\pi\le\theta<\pi} u(2;\theta) =$$

$$\max_{-\pi\le\theta<\pi}\{3\sin(2\theta)+1\} = 3\cdot 1 + 1 = \boxed{4}.$$

(b) $$u(0;0) \overset{\text{(Mean Value Property)}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\{3\sin(2\theta)+1\}d\theta = \frac{1}{2\pi}\left\{-\frac{3}{2}\cos(2\theta)+\theta\right\}\Bigg|_{-\pi}^{\pi} = \boxed{1}$$

#2 Solve $u_{xx}+u_{yy}=0$ in the disk $0 \le r < a$, $-\pi \le \theta < \pi$ with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$.

We use the solution obtained by separating variables in polar coordinates:

$$u(r;\theta) = \sum_{n=-\infty}^{\infty} c_n\left(\frac{r}{a}\right)^{|n|} e^{in\theta}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)\, e^{-in\varphi}\, d\varphi$, $(n = 0, \pm 1, \pm 2, \dots)$.

(These are the complex forms of (10), (11), and (12) p. 160.)

In our case, $h(\varphi) = 1 + 3\sin(\varphi) = 1 + \frac{3}{2i}(e^{i\varphi} - e^{-i\varphi})$.

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#2 (cont.) Therefore

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+\frac{3}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)\right]e^{-in\varphi}\,d\varphi = \begin{cases} 1 & \text{if } n=0, \\ \frac{3}{2i} & \text{if } n=1, \\ -\frac{3}{2i} & \text{if } n=-1, \\ 0 & \text{otherwise}. \end{cases}$$

Thus,

$$\begin{aligned} u(r;\theta) &= c_0 + c_1\left(\frac{r}{a}\right)e^{i\theta} + c_{-1}\left(\frac{r}{a}\right)e^{-i\theta} \\ &= 1 + \frac{r}{a}\left(\frac{3}{2i}e^{i\theta} + -\frac{3}{2i}e^{-i\theta}\right) \\ &= \boxed{1 + \frac{3r\sin(\theta)}{a}}\,. \end{aligned}$$