Mathematics 325
Supplementary Problems for Sec. 1.6

Classify the following second-order linear partial differential equations as either hyperbolic, elliptic, or parabolic. In the case of hyperbolic or parabolic equations, find the general solution in the plane.

A. $u_{xx} + 3u_{yy} - 2u_{x} + 24u_{y} + 5u = 0$

B. $u_{xx} - 3u_{xy} - 4u_{yy} = 0$

C. $u_{xx} + 9u_{yy} - 6u_{xy} = 0$

D. $u_{xx} + 4u_{yy} - 4u_{xy} + u = 0$

E. $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} + 2u_{x} - 2u_{y} = 0$

F. $u_{xx} + 2u_{xy} + 3u_{yy} - 2u_{x} + 24u_{y} + 5u = 0$

G. $u_{xx} + u_{yy} - 2u_{xy} + u_{x} - u_{y} = 0$

H. $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_{y} = 0$

I. $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} = \sin(x+y)$

(A) $u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$

$B^2 - 4AC = 0^2 - 4(1)(3) < 0$ $\boxed{\text{elliptic}}$

(B) $u_{xx} - 3u_{xy} - 4u_{yy} = 0$

$B^2 - 4AC = (-3)^2 - 4(1)(-4) > 0$ $\boxed{\text{hyperbolic}}$

Factoring the differential operator gives

(H) $\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u = 0$

Making the change of coordinates

$$\xi = \beta x - \alpha y = -4x - y$$
$$\eta = \delta x - \gamma y = x - y$$

and using the chain rule, we find that (H) is equivalent to

$$\frac{\partial^2 u}{\partial \eta \partial \xi} = 0$$

and hence $u = f(\xi) + g(\eta)$ where f and g are arbitrary C^2-functions of a single real variable. Thus

$$\boxed{u = f(4x+y) + g(x-y)}$$

is the general C^2-solution in the plane.

(C) $u_{xx} + 9u_{yy} - 6u_{xy} = 0$

$B^2 - 4AC = (-6)^2 - 4(1)(9) = 0$ $\boxed{\text{parabolic}}$

Factoring the differential operator gives

(†††) $\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)^2 u = 0.$

Making the change of variables

$$\xi = \beta x - \alpha y = -3x - y$$
$$\eta = \alpha x + \beta y = x - 3y$$

and using the chain rule, we find that (†††) is equivalent to

$$\frac{\partial^2 u}{\partial \eta^2} = 0$$

and hence $u = \eta f(\xi) + g(\xi)$ where f and g are arbitrary C^2-functions of a single real variable. Thus

$$\boxed{u = (x-3y)f(3x+y) + g(3x+y)}$$

is the general C^2-solution in the plane.

(D) $u_{xx} - 4u_{xy} + 4u_{yy} + u = 0$

$$B^2 - 4AC = (-4)^2 - 4(1)(4) = 0 \qquad \boxed{\text{parabolic}}$$

Factoring the differential operator gives

$$(\dagger) \qquad \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)u + u = 0.$$

Making the coordinate change

$$\xi = \beta x - \alpha y = -2x - y$$
$$\eta = \alpha x + \beta y = x - 2y$$

and using the chain rule we find that $(\dagger)$ is equivalent to

$$25\frac{\partial^2 u}{\partial \eta^2} + u = 0.$$

Thus $u = c_1(\xi)\cos\left(\frac{\eta}{5}\right) + c_2(\xi)\sin\left(\frac{\eta}{5}\right)$ by ordinary differential equation methods; c_1 and c_2 denote arbitrary "constants" which may vary with ξ, but are independent of η. Thus

$$\boxed{u(x,y) = f(2x+y)\cos\left(\frac{x-2y}{5}\right) + g(2x+y)\sin\left(\frac{x-2y}{5}\right)},$$

where f and g are arbitrary C^2-functions of a single real variable, is the general C^2 solution in the plane.

(E) $u_{xx} - 4u_{xy} + 3u_{yy} + 2u_x - 2u_y = 0$

$$B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0 \qquad \boxed{\text{hyperbolic}}$$

(E) (cont.) Writing the PDE in factored operator form gives

$$(*) \qquad \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + 2\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0$$

Let $\begin{cases} \xi = 3x+y \\ \eta = x+y. \end{cases}$ Then $\begin{cases} \dfrac{\xi-\eta}{2} = x \\ \dfrac{3\eta-\xi}{2} = y. \end{cases}$

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)v \Rightarrow 2\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} = -\frac{1}{2}\frac{\partial v}{\partial x} + \frac{3}{2}\frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)v \Rightarrow -2\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}$$

Therefore the PDE (*) becomes

$$\left(2\frac{\partial}{\partial \xi}\right)\left(-2\frac{\partial}{\partial \eta}\right)u + 2\left(2\frac{\partial}{\partial \xi}\right)u = 0$$

$$(\dagger) \qquad \frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{\partial u}{\partial \xi} = 0.$$

Letting $v = \dfrac{\partial u}{\partial \xi}$, $(\dagger)$ becomes

$$\frac{\partial v}{\partial \eta} - v = 0$$

$$\therefore \quad v = c_1(\xi)e^{\eta}$$

Substituting $v = \dfrac{\partial u}{\partial \xi}$ we have $\dfrac{\partial u}{\partial \xi} = c_1(\xi)e^{\eta}$. Integrating both sides with respect to ξ, holding η fixed gives

$$u = \int c_1(\xi)e^{\eta}\,d\xi = e^{\eta}f(\xi) + g(\eta) \qquad \left(f(\xi) = \int c_1(\xi)\,d\xi\right)$$

$$\boxed{\therefore \quad u(x,y) = e^{x+y}f(3x+y) + g(x+y)}$$

where f and g are arbitrary twice continuously differentiable functions of a single real variable.

(F) $u_{xx} + 2u_{xy} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$

$B^2 - 4AC = (2)^2 - 4(1)(3) = -8 < 0$ $\boxed{\text{elliptic}}$

(G) $u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0$ $B^2 - 4AC = (-2)^2 - 4(1)(1)$

$= 0$

$\boxed{\text{Parabolic}}$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

Let $\xi = x + y$ and $\eta = y - x$. Then $\frac{\xi + \eta}{2} = y$ and $\frac{\xi - \eta}{2} = x$.

If $v = v(x,y)$ is C^1 then

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} = -\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)v.$$

Therefore the PDE (e) is equivalent to

$$4\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial u}{\partial \eta} = 0$$

or

$$\frac{\partial v}{\partial \eta} - \frac{1}{2}v = 0$$

where $v = \frac{\partial u}{\partial \eta}$. An integrating factor is $e^{-\frac{\eta}{2}}$ so

$$e^{-\frac{\eta}{2}}\frac{\partial v}{\partial \eta} - \frac{1}{2}e^{-\frac{\eta}{2}}v = 0,$$

which is equivalent to

$$\frac{\partial}{\partial \eta}\left(e^{-\frac{\eta}{2}}v\right) = 0.$$

(G) (cont.) Consequently, integrating with respect to η holding ξ fixed yields

$$e^{-\eta/2} v = c_1(\xi)$$

or

$$\left(\frac{\partial u}{\partial \eta} =\right) v = c_1(\xi) e^{\eta/2} .$$

Integrating with respect to η again, holding ξ fixed, gives

$$u = 2c_1(\xi)e^{\eta/2} + c_2(\xi) .$$

Consequently, the general solution to (e) in the xy-plane is

$$\boxed{u(x,y) = f(x+y)e^{\frac{y-x}{2}} + g(x+y)}$$

where f and g are C^2-functions of a single real variable.

(H) $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_y = 0$

$$B^2 - 4AC = (-4)^2 - 4(1)(5) = -4 < 0$$

$\boxed{\text{Elliptic}}$

(I) $\quad u_{xx} - 4u_{xy} + 3u_{yy} = \sin(x+y)$

$$B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0$$

$\boxed{\text{Hyperbolic}}$

$$\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \sin(x+y)$$

Let $\xi = 3x + y$ and $\eta = x + y$. Then $\frac{\xi - \eta}{2} = x$ and $\frac{3\eta - \xi}{2} = y$.

If $v = v(x,y)$ is C^1 then

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} = +\frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)v$$

and

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} = -\frac{1}{2}\frac{\partial v}{\partial x} + \frac{3}{2}\frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)v$$

therefore the PDE (a) is equivalent to

$$-4\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = \sin(\eta)$$

Integrating with respect to η holding ξ fixed yields

$$\frac{\partial u}{\partial \xi} = +\frac{1}{4}\cos(\eta) + c_1(\xi).$$

Integrating with respect to ξ holding η fixed gives

$$u = +\frac{\xi}{4}\cos(\eta) + \int c_1(\xi)\,d\xi + c_2(\eta).$$

Therefore the general solution to (a) in the xy-plane is

$$\boxed{u(x,y) = +\frac{(3x+y)}{4}\cos(x+y) + f(3x+y) + g(x+y)}$$

where f and g are C^2-functions of a single real variable.