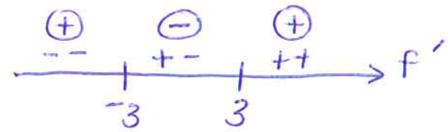
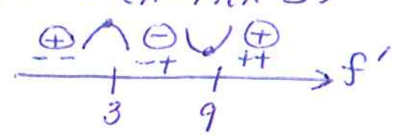
## Chapter 3, Section 1

4. Derivative is positive on $(1,3)$ and $(5,\infty)$ (increasing)
Derivative is negative on $(-\infty,1)$ and $(3,5)$ (decreasing)

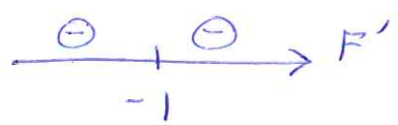
8. $f(x) = \frac{1}{3}x^3 - 9x + 2$
$f'(x) = x^2 - 9 = (x+3)(x-3)$
<u>CN</u> : $x = \pm 3$
$f$ is increasing on $(-\infty,-3) \cup (3,\infty)$
$f$ is decreasing on $(-3,3)$



12. $f(x) = 324x - 72x^2 + 4x^3$
$f'(x) = 324 - 144x + 12x^2 = 12(x^2-12x+27) = 12(x-9)(x-3) = 0$
<u>CN</u> : $x = 3, 9$
Increasing on $(-\infty,3) \cup (9,\infty)$
decreasing on $(3,9)$
If $x=3$, $y = 324(3) - 72(9) + 4(27) = 432$
If $x=9$, $y = 324(9) - 72(81) + 4(729) = 0$
relative max at $(3, 432)$, relative min at $(9,0)$



16. $F(x) = 3 - (x+1)^3$
$F'(x) = -3(x+1)^2$
<u>CN</u> : $x = -1$
If $x=-1$, $y = 3 - 0 = 3$ $(-1,3)$
The critical point $(-1,3)$ is neither a relative max nor a relative min.

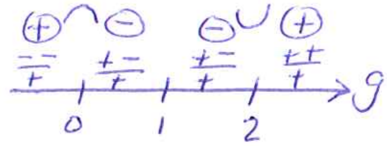


## Chapter 3, Section 1

20. $g(x) = \dfrac{x^2}{x-1}$

$g'(x) = \dfrac{2x(x-1) - x^2(1)}{(x-1)^2} = \dfrac{2x^2 - 2x - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$

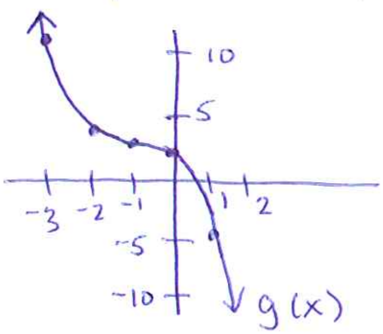
CN: $x = 0, 1, 2$




Critical points: $(0,0)$ is a max
$(2,4)$ is a min

24. $g(x) = 3 - (x+1)^3$

From #16, we know this function is always decreasing except at $x = {}^-1$, where there is a horizontal tangent.



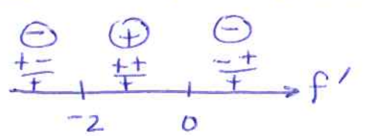

| x | f(x) |
|---|---|
| -3 | 3+8 = 11 |
| -2 | 3+1 = 4 |
| -1 | 3 |
| 0 | 3-1 = 2 |
| 1 | 3-8 = -5 |
| 2 | 3-27 = -24 |

28. $f(x) = \dfrac{x+1}{x^2+x+1}$

$f'(x) = \dfrac{(1)(x^2+x+1) - (x+1)(2x+1)}{(x^2+x+1)^2} = \dfrac{x^2+x+1 - (2x^2+3x+1)}{(x^2+x+1)^2} = \dfrac{-x^2 - 2x}{(x^2+x+1)^2}$

$= \dfrac{-x(x+2)}{(x^2+x+1)^2} = 0$

CN: $x = 0, {}^-2$ (undef if $x^2 + x + 1 = 0$, but then $x$ is complex.



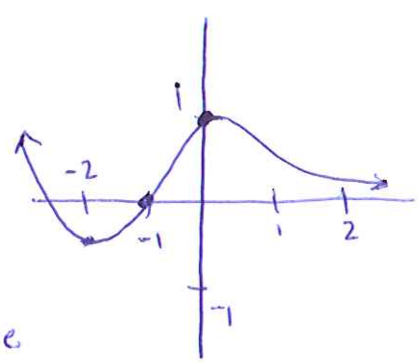

increasing on $({}^-2, 0)$

decreasing on $(-\infty, {}^-2) \cup (0, \infty)$

max at $(0, 1)$

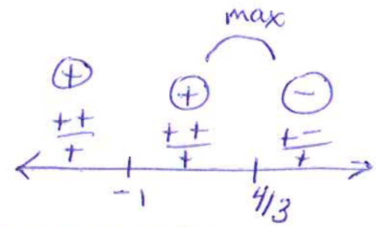
min at $({}^-2, {}^-\frac{1}{3})$

| x | f(x) |
|---|---|
| 0 | 1 |
| -1 | 0 |
| -2 | -1/3 |
| -3 | -2/7 |
| 1 | 2/3 |
| 2 | 3/7 |
| positive | positive |



## Chapter 3, Section 1

32. $f'(x) = \dfrac{(x+1)^2(4-3x)^3}{(x^2+1)^2}$




$\underline{CN}$: $x=-1$ gives neither a max nor a min

$x = 4/3$ gives a local max.

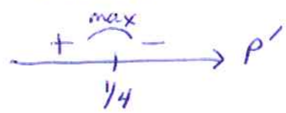
36. $C(x) = 2x^2 + 3x + 5$ , $R(x) = x \cdot p(x) = 5x - 2x^2$

Profit $= P(x) = 5x - 2x^2 - (2x^2 + 3x + 5)$

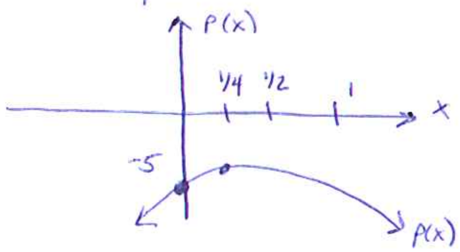
$= -4x^2 + 2x - 5$

$P'(x) = -8x + 2$

$\underline{CN}$ : $x = 1/4$




maximum profit is when $x = \frac{1}{4}$ unit is produced.



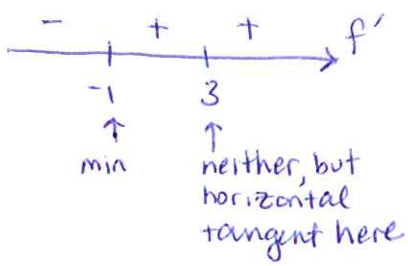

$P(1/4) = \frac{-1}{4} + \frac{1}{2} - 5 = -19/4$

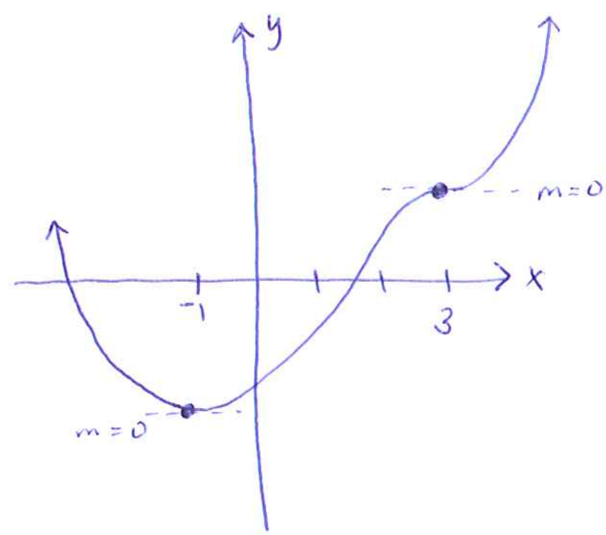
this is maximum profit -- a poor business venture!

42. $f'(x) < 0$ for $x < -1$

$f'(x) > 0$ for $x > 3$, $-1 < x < 3$

$f'(-1) = f'(3) = 0$



## Chapter 3, Section 1

43. Find $a, b, c$ so that $f(x) = ax^2 + bx + c$ has a max at $(5,12)$ and crosses the $y$-axis at $(0,3)$.

Using the points: $3 = 0 + 0 + c$, so $c = 3$

$12 = 25a + 5b + 3$

$9 = 25a + 5b$, $5b = 9 - 25a$,

$b = \frac{9}{5} - 5a$

maximum at $(5,12)$, so $f'(5) = 0$.

$f'(x) = 2ax + b$

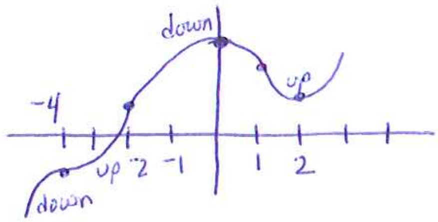
$0 = 10a + b$

$0 = 10a + \frac{9}{5} - 5a = 5a + \frac{9}{5}$

$5a = -9/5$, $a = -9/25$, $b = \frac{9}{5} - 5(-9/25) = 18/5$

so $a = -9/25$, $b = 18/5$, and $c = 3$.

# Chapter 3, Section 2

2. 


2nd deriv is positive on $(-4,-2)\cup(1,\infty)$
negative on $(-\infty,-4)\cup(-2,1)$

4. $f(x) = x^3+3x^2+1$

$f'(x) = 3x^2+6x = 3x(x+2)$

<u>CN</u>: $x = 0, -2$

$f'$: $--$ ($\oplus$) | $-2$ | $-+$ ($\ominus$) | $0$ | $++$ ($\oplus$)

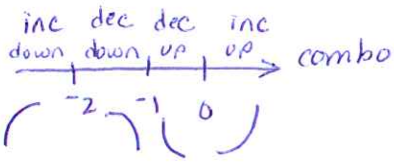
inc $(-\infty,-2)\cup(0,\infty)$, dec $(-2,0)$
max $(-2,5)$, min $(0,1)$

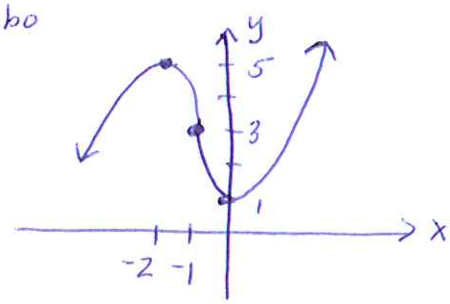
$f''(x) = 6x+6 = 6(x+1)$

<u>IN</u>: $x = -1$

$f''$: $-$ | $-1$ | $+$

conc up $(-1,\infty)$, down $(-\infty,-1)$
inflection point $(-1,3)$




8. $f(x) = x^5-5x$

$f'(x) = 5x^4-5 = 5(x^4-1) = 5(x^2+1)(x+1)(x-1)$

<u>CN</u>: $x = \pm 1$

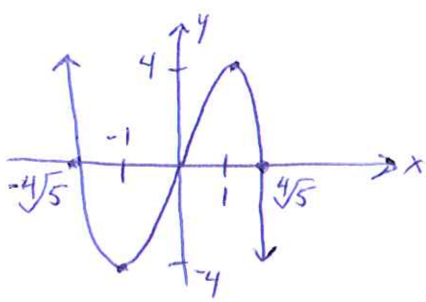
$f'$: $+--$ ($\oplus$) | $-1$ | $++-$ ($\ominus$) | $1$ | $+++$ ($\oplus$)

inc $(-\infty,-1)\cup(1,\infty)$
dec $(-1,1)$
max $(-1,4)$, min $(1,-4)$

$f''(x) = 20x^3$

$f''$: $-$ | $0$ | $+$

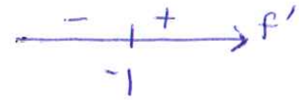
conc up $(0,\infty)$, down $(-\infty,0)$
infpt $(0,0)$



## Chapter 3, Section 2

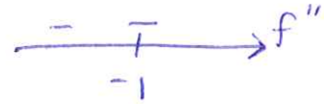
16. $f(x) = (x+1)^{2/3}$

$f'(x) = \frac{2}{3}(x+1)^{-1/3} = \frac{2}{3(x+1)^{1/3}}$

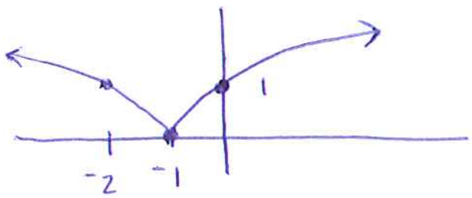



$f(-1)=0$
inc on $(-1,\infty)$
dec on $(-\infty,-1)$
min at $(-1,0)$

$f''(x) = -\frac{2}{9}(x+1)^{-4/3} = \frac{-2}{9(x+1)^{4/3}}$




conc down on $(-\infty,-1)\cup(-1,\infty)$
no inf pts.

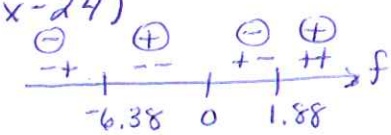



note: to sketch, plot a few extra points. Notice $f'$ DNE for $x=-1$.

20. $f(x) = x^4 + 6x^3 - 24x^2 + 24$

$f'(x) = 4x^3 + 18x^2 - 48x = 2x(2x^2 + 9x - 24)$

<u>CN</u>: $x=0$, $x = \frac{-9 \pm \sqrt{273}}{4} \approx -6.38, 1.88$



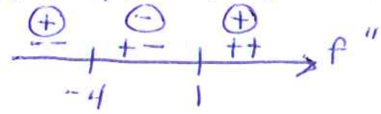

inc on $\left(\frac{-9-\sqrt{273}}{4}, 0\right) \cup \left(\frac{-9+\sqrt{273}}{4}, \infty\right)$

dec on $\left(-\infty, \frac{-9-\sqrt{273}}{4}\right) \cup \left(0, \frac{-9+\sqrt{273}}{4}\right)$

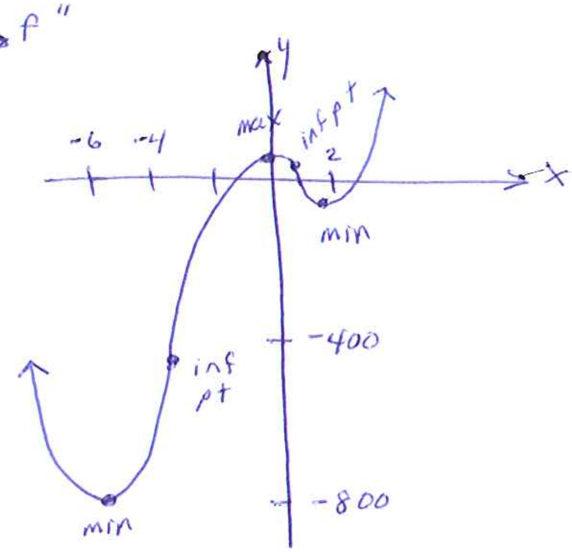
max at $(0, 24)$
min at $\left(\frac{-9-\sqrt{273}}{4}, \sim -854.22\right)$ and also at $\left(\frac{-9+\sqrt{273}}{4}, \sim -8.47\right)$

$f''(x) = 12x^2 + 36x - 48 = 12(x^2+3x-4) = 12(x+4)(x-1)$

<u>IN</u>: $x = -4, 1$




conc up on $(-\infty,-4)\cup(1,\infty)$
conc down on $(-4,1)$
Inf pts $(-4,-488)$ and $(1,7)$



# Chapter 3, Section 2

24. $f(x) = x + \frac{1}{x} = x + x^{-1}$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x+1)(x-1)}{x^2}$ CN: $x = -1, 1, 0$

$f''(x) = 2x^{-3}$ $f''(-1) = -2 < 0$ $\cap$ max

$f''(1) = 2 > 0$ $\cup$ min

$f''(0)$ undefined

$(1, 2)$ is a minimum, $(-1, -2)$ is a maximum

28. $f(x) = \left(\frac{x}{x+1}\right)^2$

$f'(x) = 2\left(\frac{x}{x+1}\right)\left(\frac{x+1-x}{(x+1)^2}\right) = \frac{2x}{(x+1)^3}$ CN: $x = 0, -1$

$f''(x) = \frac{2(x+1)3 - 2x(3)(x+1)^2(1)}{(x+1)^6} = \frac{(x+1)^2[2x+2-6x]}{(x+1)^6} = \frac{2-4x}{(x+1)^4}$

$f''(0) = 2 > 0$ $\cup$ min

$f''(-1)$ is undefined

min at $(0, 0)$

32. $h(t) = \frac{(t+3)^3}{(t-1)^2}$

$h'(t) = \frac{3(t+3)^2(t-1)^2 - (t+3)^3(2)(t-1)}{(t-1)^4} = \frac{(t+3)^2(t-1)[3(t-1)-2(t+3)]}{(t-1)^4}$

$= \frac{(t+3)^2(3t-3-2t-6)}{(t-1)^3} = \frac{(t+3)^2(t-9)}{(t-1)^3}$ CN: $x = -3, 9, 1$

$h''(t) = \frac{[2(t+3)(t-9) + (t+3)^2(1)](t-1)^3 - (t+1)^2(t-9)(3)(t-1)^2}{(t-1)^6}$

$= \frac{(t+3)(t-1)^2[[2(t-9) + (t+3)](t-1) - (t+3)(t-9)(3)]}{(t-1)^6}$

$= \frac{(t+3)[(3t-15)(t-1) - 3(t-9)(t+3)]}{(t-1)^4}$ (next pg)

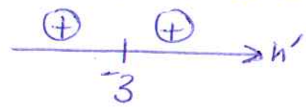
## Chapter 3, section 2

32. (cont) $h''(-3) = 0$ test fails

$$h''(9) = \frac{12[12(8) - 0]}{8^4} > 0 \quad \cup \text{ min}$$

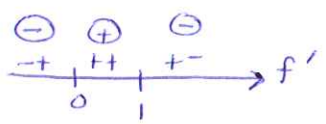
$$h''(1) = \frac{4[-12(0) - 3(-8)(4)]}{0} \quad \text{undefined, test fails}$$

Notice $x = 1$ won't give an extremum anyway, since $f(1)$ is undefined. However, we need to use $1^{st}$ deriv test on $x = -3$ to see if it's an extremum. $h'(t) = \frac{(t+3)^2(t-9)}{(t-1)^3}$



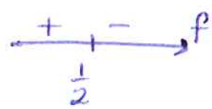

$x = -3$ does not give an extremum. The only extremum is a minimum at $(9, 27)$.

36. $f'(x) = x(1-x)$




a) inc on $(0,1)$, dec on $(-\infty, 0) \cup (1, \infty)$

$f''(x) = (1)(1-x) + (x)(-1) = 1 - 2x$

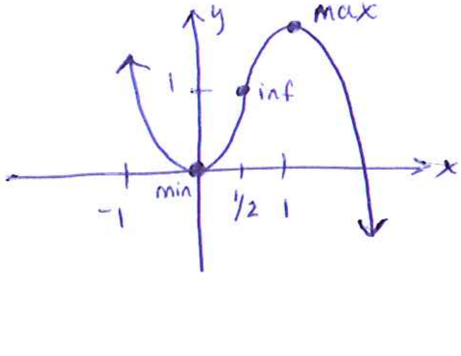



b) conc up on $(-\infty, 1/2)$
conc down on $(1/2, \infty)$

c) min when $x = 0$
max when $x = 1$
inf pt when $x = 1/2$

$\Big\}$ we don't know f, so we can't find actual points.

d)



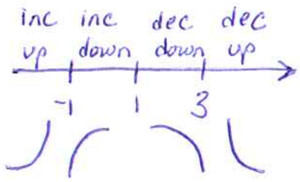
## Chapter 3, Section 2

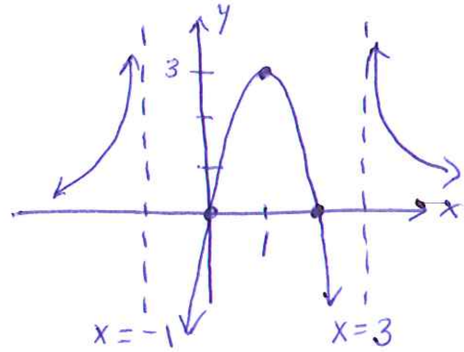
38. f is discontinuous at $x = -1$, $x = 3$

increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$ <u>except</u> at discontinuities

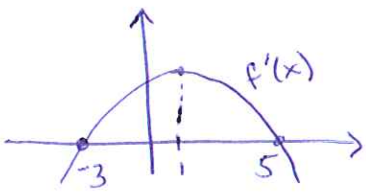
concave up on $(-\infty, -1) \cup (3, \infty)$, down on $(-1, 3)$

$f(0) = 0 = f(2)$, $f(1) = 3$



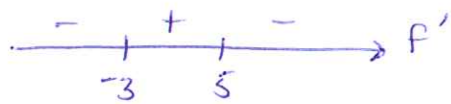




40.




$f'(x) < 0$ for $(-\infty, -3) \cup (5, \infty)$ dec

$f'(x) > 0$ for $(-3, 5)$ inc



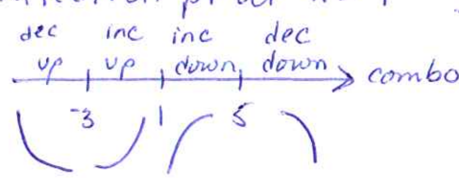

min at $x = -3$, max at $x = 5$

$f''(x) = 0$ at $x = 1$

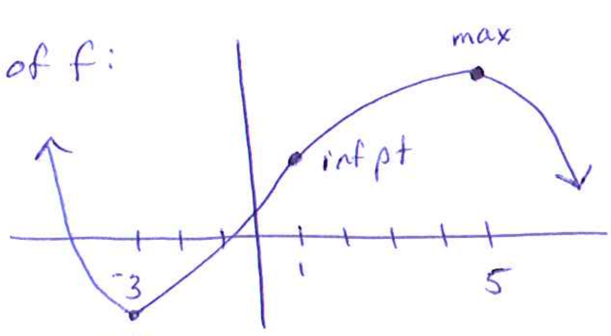
$f''(x) > 0$ on $(-\infty, 1)$ conc up

$f''(x) < 0$ on $(1, \infty)$ conc down

inflection pt at $x = 1$


possible graph of f:



## Chapter 3, Section 3

4. $\lim\limits_{x \to +\infty} (1+x^2)^3 = \infty$

6. $\lim\limits_{x \to +\infty} \dfrac{1-3x^3}{2x^3-6x+2} = -3/2$

8. $\lim\limits_{x \to +\infty} \dfrac{x^2+x-5}{1-2x-x^3} = 0$

10. $\lim\limits_{x \to \infty} \dfrac{1-2x^3}{x+1} = \lim\limits_{x \to \infty} \dfrac{-2x^3}{x} = -\infty$

12. No vertical asymptotes
Horizontal asymptote at $y=0$, since $\lim\limits_{x \to +\infty} f(x) = 0$

16. No vertical asymptotes
Horizontal asymptotes at $y=2$ and $y=-3$.

20. $f(t) = \dfrac{t+2}{t^2}$ vertical asymptote at $t=0$.
$\lim\limits_{t \to \infty} \dfrac{t+2}{t^2} = \lim\limits_{t \to \infty} \dfrac{t}{t^2} = 0$ Horizontal asymptote at $y=0$.

22. $g(x) = \dfrac{5x^2}{x^2-3x-4} = \dfrac{5x^2}{(x-4)(x+1)}$ vertical asymptotes $x=4,-1$
$\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to \infty} \dfrac{5x^2}{x^2} = 5$ horizontal asymptote $y=5$

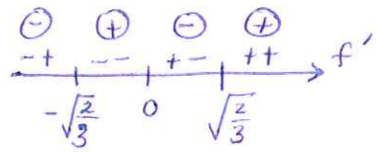
24. $g(t) = \dfrac{t}{\sqrt{t^2-4}} = \dfrac{t}{\sqrt{(t+2)(t-2)}}$ vertical asymptotes $t = \pm 2$
$\lim\limits_{t \to \infty} g(t) = \lim\limits_{t \to \infty} \dfrac{t}{\sqrt{t^2}} = \lim\limits_{t \to \infty} \dfrac{t}{|t|} = \lim\limits_{t \to \infty} \dfrac{t}{t} = 1$ HA $t=1$
$\lim\limits_{t \to -\infty} g(t) = \lim\limits_{t \to -\infty} \dfrac{t}{|t|} = \lim\limits_{t \to -\infty} \dfrac{t}{-t} = -1$ HA $t=-1$

## Chapter 3, Section 3

28. $f(t) = 3t^4 - 4t^2 + 3$

No HA or VA. Notice as $t \to \pm\infty$, $f(t) \to \infty$ (up on both ends)

$f'(t) = 12t^3 - 8t = 4t(3t^2 - 2)$




<u>CN</u>: $t = 0, \pm\sqrt{2/3}$

inc $\left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

dec $\left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(0, \sqrt{\frac{2}{3}}\right)$

min $\left(\pm\sqrt{\frac{2}{3}}, \frac{5}{3}\right)$, max $(0,3)$

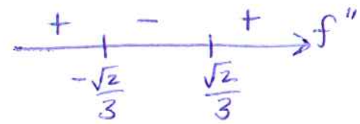
$f''(t) = 36t^2 - 8 = 4(9t^2 - 2)$

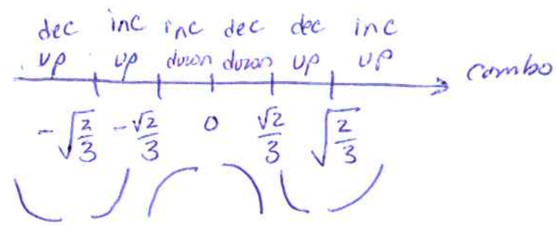
<u>IN</u>: $t = \pm\sqrt{2}/3$

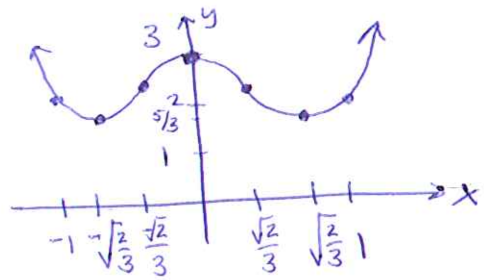
conc up $\left(-\infty, -\frac{\sqrt{2}}{3}\right) \cup \left(\frac{\sqrt{2}}{3}, \infty\right)$

conc down $\left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$

inf pts $\left(\pm\frac{\sqrt{2}}{3}, \frac{61}{27}\right)$

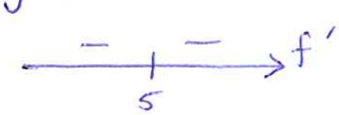


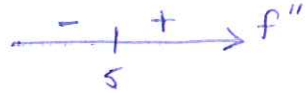

32. $f(x) = \dfrac{x+3}{x-5}$  <u>VA</u>: $x = 5$, <u>HA</u>: $y = 1$

$f'(x) = \dfrac{(x-5)-(x+3)}{(x-5)^2} = \dfrac{-8}{(x-5)^2}$

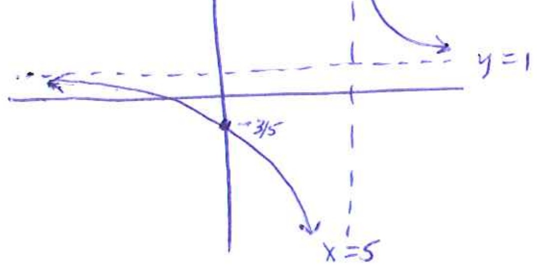


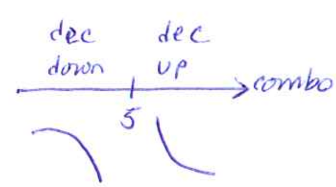

never inc.
dec $(-\infty, 5) \cup (5, \infty)$
no extrema

$f''(x) = 16(x-5)^{-3}$




conc up on $(5, \infty)$
conc down on $(-\infty, 5)$
no inf pts, since $f(5)$ is undefined.



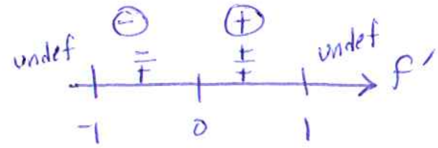



## Chapter 3, Section 3

36. $f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$ Notice domain is $-1 < x < 1$.

VA: $x = \pm 1$, HA: $y = 0$

$f'(x) = \frac{-1}{2}(1-x^2)^{-3/2}(-2x) = \frac{x}{(1-x^2)^{3/2}}$ CN: $x = 0, \pm 1$



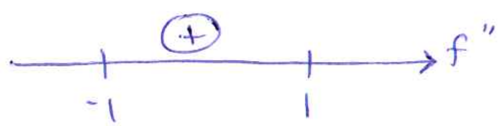

dec on $(-1,0)$

inc on $(0,1)$

min at $(0,1)$

$$f''(x) = \frac{(1)(1-x^2)^{3/2} - (x)(\frac{3}{2})(1-x^2)^{1/2}(-2x)}{(1-x^2)^3}$$

$$= \frac{(1-x^2)^{1/2}[1-x^2+3x^2]}{(1-x^2)^3} = \frac{1+2x^2}{(1-x^2)^{5/2}}$$ IN: $x = \pm 1$




concave up on $(-1,1)$

no inf. pts.