Section 4.2

4. $\ln \sqrt{e} = \ln e^{1/2} = 1/2$

6. $e^{2\ln 3} = e^{\ln 3^2} = 3^2 = 9$

8. $\ln \frac{e^3\sqrt{e}}{e^{1/3}} = \ln e^3 + \ln\sqrt{e} - \ln e^{1/3} = 3 + \frac{1}{2} - \frac{1}{3} = 19/6$

11.
$$3 = 2 + 5e^{-4x}$$
$$1 = 5e^{-4x}$$
$$\frac{1}{5} = e^{-4x}$$
$$\ln\frac{1}{5} = e^{-4x}$$
$$\ln\frac{1}{5} = -4x$$
$$-\frac{1}{4}\ln\frac{1}{5} = x$$
$$x = \frac{\ln(\frac{1}{5})^{-1}}{4} = \frac{\ln 5}{4}$$
$$x \approx 0.402$$

12.
$$-2\ln x = b$$
$$\ln x = -\frac{1}{2}b$$
$$x = e^{-b/2}$$

14.
$$5 = 3\ln x - \frac{1}{2}\ln x$$
$$5 = (3 - \frac{1}{2})\ln x = \frac{5}{2}\ln x$$
$$\cancel{5}(\frac{2}{\cancel{5}}) = \ln x$$
$$e^2 = x$$

16.
$$\ln x = 2(\ln 3 - \ln 5) = 2\ln 3/5 = \ln\frac{9}{25}$$
$$x = 9/25$$

23.
$$\log_5(2x) = 7 = \frac{\ln 2x}{\ln 5}, \text{ so } 7\ln 5 = \ln 2x$$
$$\log_5 2 + \log_5$$
$$= \ln 2 + \ln x.$$
$$7\ln 5 - \ln 2 = \ln x.$$
$$\ln x \approx 10.5729$$

Section 4.2 - continued

26\. Find $\frac{1}{a}\ln\left(\frac{\sqrt{b}}{c}\right)^a$ if $\ln b = 6$ and $\ln c = {}^-2$

$$\frac{1}{a}\ln\left(\frac{\sqrt{b}}{c}\right)^a = \frac{1}{a}(a)\ln\left(\frac{\sqrt{b}}{c}\right)$$
$$= \ln b^{1/2} - \ln c$$
$$= \frac{1}{2}\ln b - \ln c = \frac{1}{2}(6) - ({}^-2) = 5.$$

28\. Annual rate 7% comp. continuously. How long for money to double?
(find t so that $B = 2P$)

$$B = Pe^{rt}$$
$$2\cancel{P} = \cancel{P}e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.902$$

32\. x thousand complimentary copies will give sales of $f(x) = 20 - 15e^{-0.02x}$ thousand copies. Find x so that $f(x) = 12$ thousand.

$$12 = 20 - 15e^{-0.02x}$$
$$15e^{-0.02x} = 8 \longrightarrow e^{-0.02x} = \frac{8}{15}$$
$$-0.02x = \ln\frac{8}{15} \longrightarrow x = \frac{\ln 8/15}{-0.02} \approx 31.43$$

So give away 31,430 copies.

39\. Originally, $t=0$, $Q = Q_0 \longrightarrow Q_0 = Q_0 e^{0 \cdot k} \longrightarrow Q_0 = Q_0$.

when $t = 5730$, $Q = \frac{1}{2}Q_0$ (half-life of ^{14}C, see text),

so $\frac{1}{2}\cancel{Q_0} = \cancel{Q_0}e^{5730k}$

$$\ln\frac{1}{2} = 5730k$$
$$k \approx -0.00012097.$$

currently, $t = ?$ and $Q = 0.997Q_0$, so $0.997\cancel{Q_0} = \cancel{Q_0}e^{-0.00012097t}$

$$\ln 0.997 = -0.00012097t$$
$$t \approx 24.84 \text{ years}$$ (actual age of painting)

MORE →

Section 4.2- (continued)

39. (cont).

If painting was originally done in 1640:

1640	$t=0$	$Q=Q_0$
	$t=5730$	$Q=\frac{1}{2}Q_0 \longrightarrow k \approx -0.00012097$ (just like before)
2002 ↑ now	$t=362$	$Q=?$

$$Q = Q_0 e^{-0.00012097(362)}$$
$$Q \approx 0.9572\,Q_0$$

Approximately 95.7% of the original amount of ^{14}C would remain today if the painting were original.

Section 4.3

2. $f(x) = 3e^{4x+1}$ $\qquad$ $f'(x) = 3(e^{4x+1})(4) = 12e^{4x+1}$

6. $f(x) = x^2e^x$ $\qquad$ $f'(x) = 2xe^x + x^2e^x$

8. $f(x) = xe^{-x^2}$ $\qquad$ $f'(x) = (1)(e^{-x^2}) + (x)(e^{-x^2})(-2x)$

$$= e^{-x^2} - 2x^2e^{-x^2}$$

10. $f(x) = \sqrt{1+e^x}$

$$= (1+e^x)^{1/2}$$

$f'(x) = \frac{1}{2}(1+e^x)^{-1/2}(e^x)$

11. $f(x) = e^{\sqrt{3x}}$

$$= e^{\sqrt{3}x^{1/2}}$$

$f'(x) = (e^{\sqrt{3x}})(\frac{\sqrt{3}}{2}x^{-1/2})$

$$= \frac{\sqrt{3}}{2\sqrt{x}} e^{\sqrt{3x}}$$

16. $f(x) = x \ln\sqrt{x}$

$$= x \ln(x^{1/2})$$

$f'(x) = (1)(\ln\sqrt{x}) + (x)(\frac{1}{x^{1/2}})(\frac{1}{2}x^{-1/2})$

$$= \ln\sqrt{x} + x(\frac{1}{\sqrt{x}})(\frac{1}{2})(\frac{1}{\sqrt{x}})$$

$$= \ln\sqrt{x} + \frac{1}{2}$$

<u>another way</u>:

$f(x) = x \cdot \frac{1}{2}\ln x$

$$= \frac{1}{2}x\ln x$$

$f'(x) = \frac{1}{2}\ln x + \frac{1}{2}x(\frac{1}{x})$

$$= \ln\sqrt{x} + \frac{1}{2}$$

19. $f(x) = \ln\left(\frac{x+1}{x-1}\right)$ $\qquad$ $f'(x) = \left(\frac{1}{\frac{x+1}{x-1}}\right)\left[\frac{(1)(x-1)-(x+1)(1)}{(x-1)^2}\right]$

$$= \left(\frac{\cancel{x-1}}{x+1}\right)\left(\frac{x-1-x-1}{(x-1)^{\cancel{2}}}\right)$$

$$= \frac{-2}{(x+1)(x-1)} = \frac{-2}{x^2-1}$$

Section 4.3 - (continued)

20. $f(x) = e^x \ln x$ $\qquad f'(x) = (e^x)(\ln x) + (e^x)(\frac{1}{x})$
$= e^x \ln x + \frac{e^x}{x}$.

22. $f(x) = (x+1)e^{-2x}$. Find tangent line where $x=0$.
$f'(x) = (1)e^{-2x} + (x+1)(-2)(e^{-2x})$.
$f'(0) = \text{slope} = (1) + (-2)(1) = -1$.
$m = -1$. If $x=0$, need $y = f(0) = (1)e^0 = 1$. $(0,1)$ is point.
Line: $y - 1 = -1(x-0)$
$y - 1 = -x$
$y = -x + 1$

34. $C(x) = x^3 + 20$, $p(x) = 2e^{-2x}$.
a) marginal cost: $C'(x) = 3x^2$
b) Avg unit cost: $\frac{C(x)}{x} = x^2 + \frac{20}{x}$
marginal avg cost: $\left[\frac{C(x)}{x}\right]' = 2x - 20x^{-2}$
c) Revenue: $R(x) = x\,p(x) = 2xe^{-2x}$
Marginal revenue: $R'(x) = 2e^{-2x} + 2x(-2e^{-2x})$
$= 2e^{-2x} - 4xe^{-2x}$
d) find x so that $R' = C'$: $3x^2 = 2e^{-2x} - 4xe^{-2x}$
$0 = 2e^{-2x}(1-2x) - 3x^2$
To find x, plot graph on calculator, see where x-int. is.
$x \approx 0.33522$
e) find x so that $C' = \frac{C(x)}{x}$: $3x^2 = x^2 + \frac{20}{x}$
$2x^2 = \frac{20}{x}$
$x^3 = 10$
$x = \sqrt[3]{10}$

Section 4.3 -(cont)

41. x thousand complimentary copies means $f(x) = 20 - 15e^{-0.2x}$ thousand copies of sold books. Current plan is $x = 10$.

a) If x increases by 1, $f(x)$ will increase by approx $f'(10)$.

$$f'(x) = -15(-0.2)e^{-0.2x} = 3e^{-0.2x}$$

$$f'(10) = 3e^{-2} \doteq 0.406$$

Sales will increase by approx 0.406 thousand copies, ie 406 copies.

b) Actual increase is $f(11) - f(10)$

$$= (\cancel{20} - 15e^{-0.2(11)}) - (\cancel{20} - 15e^{-0.2(10)})$$

$$\approx -1.66205 + 2.03003$$

$$\approx 0.368 \text{ thousand, ie, 368 copies.}$$

42. $D(p) = 3000e^{-0.01p}$ monthly. Monthly expenditure is $R(p) = pD(p)$.

$R(p) = 3000p\,e^{-0.01p}$.

Find p to maximize $R(p)$.

$$R'(p) = 3000e^{-0.01p} + (3000p)(-0.01)e^{-0.01p}$$

$$= \underbrace{3000e^{-0.01p}}_{\text{never } 0}\underbrace{\left(1 + \tfrac{-1}{100}p\right)}_{p=100 \text{ is crit } \#.} = 0$$

Is $p = 100$ a max or min?

$$R''(p) = 3000(-0.01)e^{-0.01p}\left(1 - \tfrac{p}{100}\right) + 3000e^{-0.01p}\left(\tfrac{-1}{100}\right)$$

$$R''(100) = -30e^{-1}(1-1) + 3000e^{-1}\left(\tfrac{-1}{100}\right)$$

$$= 0 - 30e^{-1} < 0 \text{ so max.}$$

$p = 100$ will maximize expenditure R.

Section 4.4 - Additional Exponential Models

4. $f(t) = 3 - 2e^t$

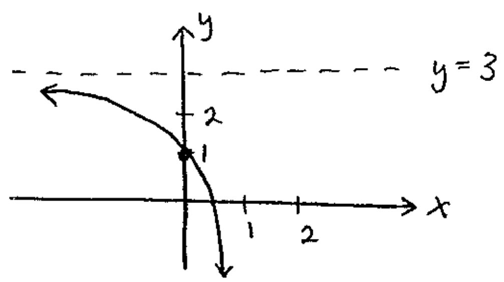
$f'(t) = -2e^t$. No critical numbers, $f'(t) < 0$ always. dec $(-\infty, \infty)$.

$f''(t) = -2e^t$. $f''(t) < 0$ always. concave down $(-\infty, \infty)$.

no maxima, minima, or inflection points.

Notice that if $t \to \infty$, $f(t) \to 3 - (\text{large} +) \to -\infty$. (right side)

Also, if $t \to -\infty$, $f(t) \to 3 - 0 = 3$. Asymp $y = 3$, happens on left.



8. $f(x) = 3 - 5e^{-x}$

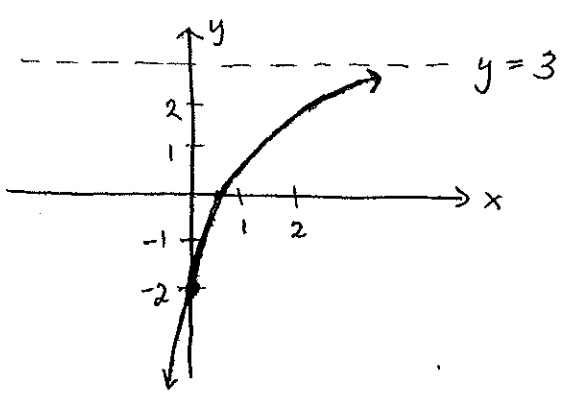
$f'(x) = 5e^{-x}$. No critical numbers. $f'(x) > 0$ always. inc $(-\infty, \infty)$

$f''(x) = -5e^{-x}$. $f''(x) < 0$ always. concave down $(-\infty, \infty)$.

no maxima, minima, or inflection points.

Notice that if $x \to \infty$, $f(x) \to 3 - 0 = 3$. Asymp $y = 3$, on right

If $x \to -\infty$, $f(x) \to 3 - (\text{large} +) \to -\infty$ (left side).



Section 4.4 - (cont)

10. $h(t) = \dfrac{2}{1+3e^{2t}}$.

$h'(t) = \dfrac{0 - 2(6e^{2t})}{(1+3e^{2t})^2} = \dfrac{-12e^{2t}}{(1+3e^{2t})^2}$ Numerator never zero, denom. never zero, no crit #'s. $h'(t) < 0$ always. decr. $(-\infty, \infty)$.

$h''(t) = \dfrac{-24e^{2t}(1+3e^{2t})^{\cancel{2}} + 12e^{2t}(2)(1+\cancel{3e^{2t}})(6e^{2t})}{(1+3e^{2t})^{\cancel{4}\,3}}$

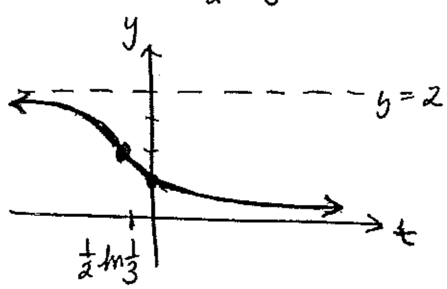
$= \dfrac{-24e^{2t} - 72e^{4t} + 144e^{4t}}{(1+3e^{2t})^3} = \dfrac{72e^{4t} - 24e^{2t}}{(1+3e^{2t})^3} = \dfrac{24e^{2t}(3e^{2t}-1)}{(1+3e^{2t})^3}$

$h''(t) = 0$ when numerator is zero (notice denom never zero). So

$3e^{2t} - 1 = 0$, so $e^{2t} = 1/3$, $2t = \ln 1/3$, $t = \frac{1}{2}\ln 1/3 \approx -0.549$

$-$ | $+$ → h'' ; $\frac{1}{2}\ln\frac{1}{3}$

Concave up on $(\frac{1}{2}\ln\frac{1}{3}, \infty)$
Concave down on $(-\infty, \frac{1}{2}\ln\frac{1}{3})$.



Notice that if $t \to -\infty$, $h(t) \to 2$.
if $t \to \infty$, $h(t) \to 0$.
Also $h(0) = 2/_{1+3} = 1/2$

18. $f(x) = x - \ln x$ for $x > 0$.

$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$ crit #'s $x = 1, 0$

undef min
$+$ | $-$ | $+$ → f' ; 0, 1

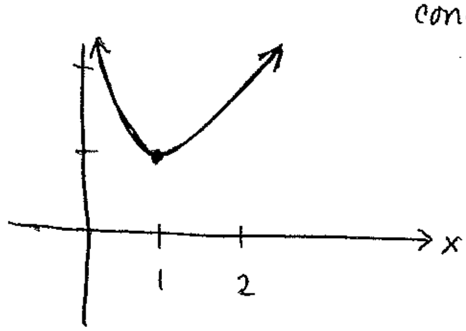
incr $(\cancel{-\infty, 0}) \cup (1, \infty)$
decr $(0, 1)$
min $(1, 1)$.

notice f is only defined for $x > 0$.

$f''(x) = 0 - (-x^{-2}) = \frac{1}{x^2}$

$+$ | $+$ → f'' ; 0

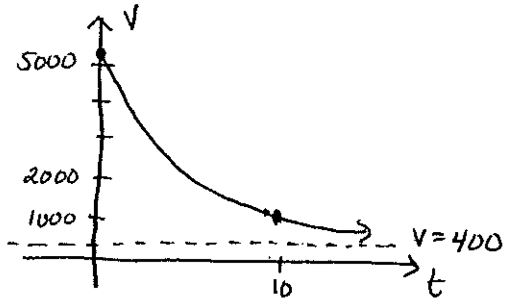
no inf pts.
conc. up on $(\cancel{-\infty, 0}) \cup (0, \infty)$



Section 4.4 - (cont)

22. $V(t) = 4800\, e^{-t/5} + 400$

a)



$V'(t) = -960\, e^{-t/5} < 0$, decr $(-\infty, \infty)$

$V''(t) = 192\, e^{-t/5} > 0$, conc up $(-\infty, \infty)$.

As $t \to \infty$, $V \to 400$

b) $V(0) = 4800(1) + 400 = 5200$

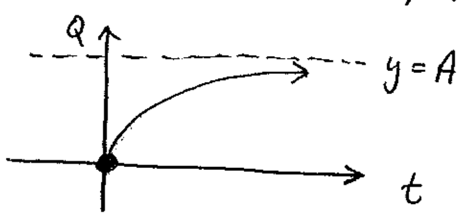
c) $V(10) = 4800\, e^{-2} + 400 \approx 1049.61$

26. $Q(t) = A(1 - e^{-kt})$, $k > 0$, $A \geq 0$.

$Q'(t) = A(ke^{-kt}) > 0$, so increasing $(-\infty, \infty)$

$Q''(t) = -Ak^2 e^{-kt} < 0$, so concave down $(-\infty, \infty)$

a)



$Q(0) = 0$

b) As $t \to \infty$, $Q \to A$, since over time more and more facts will be recalled. Since there are only A relevant facts, this is the most that can be recalled and serves as an upper limit for Q.

30. When x thousand are employed, profit is $P(x) = 10 + \ln\left(\frac{x}{25}\right) - 12x^2$ million dollars for $x > 0$.

$$P'(x) = \frac{1}{\left(\frac{x}{25}\right)}\left(\frac{1}{25}\right) - 24x = \frac{25}{x} \cdot \frac{1}{25} - 24x = \frac{1}{x} - 24x = 0$$

$24x = \frac{1}{x}$, $x^2 = \frac{1}{24}$, so $x \approx 0.2041241$ $\xrightarrow[0.2]{+ \quad \frown \quad -}$ P' max

Maximum profit is $P(x) = 4.692097$ million.

Profit $\approx \$4{,}692{,}097$ when $x = 204$

Section 4.4- (cont)

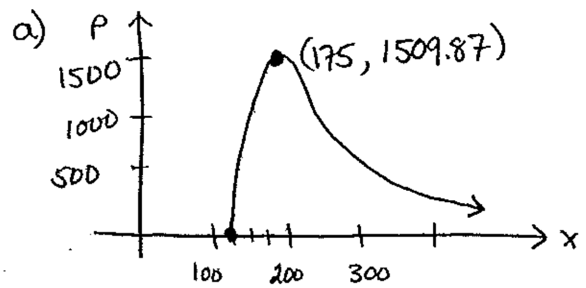
32. When $t=0$, $f(t)=\frac{B}{10}$. After t hours, $f(t)=\frac{B}{1+Ce^{-kt}}$ people know.
When $t=2$, $f(t)=B/4$. When will $f(t)=B/2$?

① $f(0)=\frac{B}{10}=\frac{B}{1+C}$ so $C=9$.

② $f(2)=\frac{B}{4}=\frac{B}{1+9e^{-k2}}$ so $9e^{-kt}=3$, $3=e^{k(2)}$
$\ln 3 = k(2)$
$k=\frac{1}{2}\ln 3$.

③ $f(t)=\frac{B}{2}=\frac{B}{1+9e^{(-\frac{1}{2}\ln 3)t}}$ so $9e^{-\frac{1}{2}t\ln 3}=1$
$9=e^{\frac{1}{2}t\ln 3}=\cancel{e}^{\cancel{\ln}}(3)^{1/2 t}$
$3^2=3^{\frac{1}{2}t}$
$t=4$ hours.

35. $P(x)=\text{rev}-\text{cost}=1000xe^{-0.02x}-125(1000e^{-0.02x})$.
$P(x)=1000e^{-0.02x}(x-125)$

a) 

$P(125)=0$ (you'll charge at least \$125 to cover cost!)

b) $P'(x)=1000(-0.02e^{-0.02x})(x-125)+1000e^{-0.02x}(1)$
$=1000e^{-0.02x}-20e^{-0.02x}(x-125)$
$=20e^{-0.02x}(50-x+125)$
$=20e^{-0.02x}(175-x)$ $\quad x=175$

max
+ | −
175

$x=\$175$ will maximize profit.

$(P(175)\approx \$1509.87)$

Section 4.4-(cont)

38. $V(t) = 200e^{\sqrt{2t}}$ dollars is value t years from now. Constant interest rate of 6% per year compounded continuously. When should you sell?

Sell when present value is greatest. Present value is

$$P(t) = V(t)e^{-0.06t} = 200e^{\sqrt{2}t^{1/2} - 0.06t}$$

$$P'(t) = 200e^{\sqrt{2}t^{1/2} - 0.06t}\left(\frac{\sqrt{2}}{2}t^{-1/2} - 0.06\right) = 0$$

$$\frac{\sqrt{2}}{2\sqrt{t}} = 0.06$$

$$\frac{\sqrt{2}}{2(0.06)} = \sqrt{t}$$

$$\frac{2}{0.0144} = t \approx 138.89 \text{ years from now}$$