7.3 - Optimizing Functions of Two Variables

2. $f(x,y) = 2x^2 - 3y^2$

$f_x = 4x = 0$, $f_y = -6y = 0$ $\Big\}$ $x=0, y=0$ crit pt $(0,0)$

$f_{xx} = 4$, $f_{yy} = -6$, $f_{xy} = 0$.

$D = (4)(-6) - (0)^2 = -24$

$D(0,0) = -24 < 0$, so $(0,0)$ is a saddle point.

6. $f(x,y) = xy + 8x^{-1} + 8y^{-1}$

$f_x = y - 8x^{-2} = 0 \longrightarrow y = \frac{8}{x^2}$. Sub in f_y: $x - \frac{8}{(8/x^2)^2} = 0$

$f_y = x - 8y^{-2} = 0$

$x = \frac{8x^4}{8^2} = \frac{x^4}{8}$

$f_{xx} = 16x^{-3}$

$8x - x^4 = 0$

$f_{yy} = 16y^{-3}$

$x(8 - x^3) = 0$

$f_{xy} = 1$

$x = 0$, $x = 2$.

y undef, $y = \frac{8}{4} = 2$.

crit pt $(2,2)$.

$D = (16x^{-3})(16y^{-3}) - 1^2$

$D(2,2) = \frac{16}{8} \cdot \frac{16}{8} - 1 = 4 - 1 = 3 > 0$. extremum!!

$f_{xx}(2,2) = \frac{16}{8} = 2 > 0$, so there is a minimum at $(2,2)$

10. $f(x,y) = -x^4 - 32x + y^3 - 12y + 7$

$f_x = -4x^3 - 32 = -4(x^3 + 8) = 0 \longrightarrow x = -2$.

$f_y = 3y^2 - 12 = 3(y^2 - 4) = 0 \longrightarrow y = \pm 2$.

$f_{xx} = -12x^2$

crit pts $(-2,2)$, $(-2,-2)$

$f_{yy} = 6y$

$f_{xy} = 0$

$D(-2,2) = -72(4)(2) < 0$ saddle at $(-2,2)$

$D(x,y) = -72x^2y - 0$

$D(-2,-2) = -72(4)(-2) > 0$

$f_{xx}(-2,-2) = -12(4) < 0$ max at $(-2,-2)$

Section 7.3- cont

14. $f(x,y) = \dfrac{x}{x^2+y^2+4}$

$f_x = \dfrac{x^2+y^2+4 - x(2x)}{(x^2+y^2+4)^2} = \dfrac{-x^2+y^2+4}{(x^2+y^2+4)^2} = 0 \longrightarrow -x^2+y^2+4=0$

$f_y = \dfrac{0-x(2y)}{(x^2+y^2+4)^2} = 0 \longrightarrow -2xy=0$, so either $x=0$ or $y=0$.

If $x=0$, fill in, $0+y^2+4=0$, no y-values.
If $y=0$, fill in, $-x^2+0+4=0$, $x=\pm 2$. Critical Points : $(\pm 2, 0)$

$f_{xx} = \dfrac{(-2x)(x^2+y^2+4)^2 - (-x^2+y^2+4)(2)(x^2+y^2+4)(2x)}{(x^2+y^2+4)^4}$

$f_{yy} = \dfrac{(-2x)(x^2+y^2+4)^2 + (2xy)(2)(x^2+y^2+4)(2y)}{(x^2+y^2+4)^4}$

$f_{xy} = \dfrac{(2y)(x^2+y^2+4)^2 - (-x^2+y^2+4)(2)(x^2+y^2+4)(2y)}{(x^2+y^2+4)^4}$

$f_{xx}(2,0) = \dfrac{(-4)(8)^2-(-4+4)(2)(8)(4)}{(8)^4} = \dfrac{-4}{8^2} = \dfrac{-4}{64} = \dfrac{-1}{16}$

$f_{yy}(2,0) = \dfrac{(-4)(8)^2-0}{(8)^4} = \dfrac{-4}{8^2} = \dfrac{-4}{64} = \dfrac{-1}{16}$

$f_{xy}(2,0) = \dfrac{0-0}{8^4} = 0$

$D(2,0) = f_{xx}(2,0)\,f_{yy}(2,0) - [f_{xy}(2,0)]^2 = \left(\frac{-1}{16}\right)\left(\frac{-1}{16}\right) - 0^2 = \frac{1}{16\cdot 16} > 0$

Since $f_{xx}(2,0) = \frac{-1}{16} < 0$, (concave down), maximum at $(2,0)$.

$f_{xx}(-2,0) = \dfrac{(4)(8)^2-(-4+4)(2)(8)(-4)}{8^4} = \dfrac{4(8)^2}{8^4} = \dfrac{1}{16}$

$f_{yy}(-2,0) = \dfrac{(4)(8)^2+0}{8^4} = \dfrac{1}{16}$

$f_{xy}(-2,0) = \dfrac{0-0}{8^4} = 0$

$D(-2,0) = \left(\frac{1}{16}\right)\left(\frac{1}{16}\right) - 0 > 0$

Since $f_{xx}(-2,0) > 0$ (concave up), minimum at $(-2,0)$.

7.3 - cont.

22. Profit $= 100x(40-8x+5y) + 100y(50+9x-7y)$
$-1000(40-8x+5y) - 3000(50+9x-7y)$

$P = 4000x - 800x^2 + 500xy + \cancel{5000y} + 900xy - 700y^2$
$-40000 + 8000x - \cancel{5000y} - 150000 - 27000x + 21000y$

$P = -15000x - 800x^2 + 1400xy + 21000y - 700y^2 - 190000$

$\partial P/\partial x = -15000 - 1600x + 1400y = 0$

$\partial P/\partial y = 1400x + 21000 - 1400y = 0$

$-200x + 6000 = 0$

$x = 30$. $1400(30) + 21000 = 1400y$

$45 = y$

critical point $(30, 45)$

$P_{xx} = -1600$
$P_{yy} = -1400$
$P_{xy} = 1400$

$D(x,y) = (-1600)(-1400) - (1400)^2 = 280,000.$

$D(30,45) = 280,000 > 0$

$P_{xx} < 0$, so profit is maximized when $x = 30$, $y = 45$. (\$3000 and \$4500)

26. Profit $= x(20-5x) + y(4-2y) - (2xy+4)$

$P = 20x - 5x^2 + 4y - 2y^2 - 2xy - 4$

$P = 20x - 5x^2 + 4y - 2y^2 - 2xy - 4$

$P_x = 20 - 10x - 2y = 0 \rightarrow y = 10 - 5x \rightarrow 4 - 4(10-5x) - 2x = 0$

$P_y = 4 - 4y - 2x = 0$

$4 - 40 + 20x - 2x = 0$

$18x = 36$

$x = 2$, $y = 0$

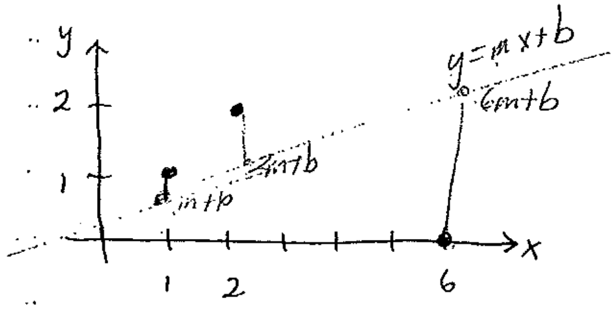
$P_{xx} = -10$
$P_{yy} = -4$
$P_{xy} = -2$

$D(x,y) = 40 - 4 = 36 > 0$

$P_{xx} = -10 < 0$ so $(2,0)$ is a max.

7.3 - cont

36. $(1,1),\ (2,2),\ (6,0)$ $\qquad y = mx + b$



$$S = (m+b-1)^2 + (2m+b-2)^2 + (6m+b)^2$$

$$S = (\cancel{m^2} \cancel{+mb} \cancel{-m} \cancel{+mb} \cancel{+b^2} \cancel{-b} \cancel{-m} \cancel{-b} \cancel{+1}) + (\cancel{4m^2} \cancel{+2mb} \cancel{-4m} \cancel{+2mb} \cancel{+b^2} - 2b \cancel{-4m} \cancel{-2b} \cancel{+4}) + (\cancel{36m^2} + \cancel{6mb} + \cancel{6mb} + \cancel{b^2})$$

$$S = 41m^2 + 18mb - 10m + 3b^2 - 6b + 5$$

$$S_m = 82m + 18b - 10 = 0 \rightarrow 41m + 9b - 5 = 0$$

$$S_b = 18m + 6b - 6 = 0 \rightarrow 9m + 3b - 3 = 0 \quad \Big\rangle\ 14m + 4 = 0$$

$$\times 3 \qquad 27m + 9b - 9 = 0$$

$$m = -4/14 = -\frac{2}{7} \longrightarrow \frac{-18/7 - 3}{-3} = b = \frac{-39/7}{-3} = \frac{13}{7}$$

$$\left.\begin{aligned} S_{mm} &= 82 \\ S_{bb} &= 6 \\ S_{mb} &= 18 \end{aligned}\right\} \quad D = 82(6) - 18^2 = 168 > 0$$

$S_{mm} = 82 > 0$, so we have a minimum for the sum of squares of distances.

The line is: $y = -\frac{2}{7}x + 13/7$